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DIFFERENTIAL SYSTEMS

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To

D. M.

PREFACE

The primary purpose of this book is to develop the theory of systems of partial differential equations and that of pfaffian systems so as to exhibit clearly the relation between the two theories. The questions treated concern almost exclusively the existence of solutions and methods of approximating them rather than their properties, whose study seems to belong to the theory of functions.

In writing the book the author has been guided by a desire for generality in results and conciseness in subject matter and proofs. As a consequence, the postulational method seemed to force itself upon him. Roughly, the plan has been to take a few existence theorems as postulates and construct the theory upon them. A consistency proof is included by proving the postulates in particular cases. The original plan included extensions of the consistency proofs, but the pressure of other duties prevented carrying this out.

The ideas and nomenclature of modern algebra, as developed, for instance, in van der Waerden's admirable treatise, have been freely used. Some modifications of certain topics, essential for our purposes, have been included, but no systematic development of the theory of commutative polynomial rings has been made. On the other hand, the theory of a certain non-commutative polynomial ring, called here a Grassmann ring, is developed in detail from the postulates in Chapter III, which together with Chapter IV develops ideas introduced by Grassmann and brought to such a high degree of perfection by Cartan. A combination of Cartan's notation, the tensor calculus, and modern algebraic concepts seems very effective. Incidentally, the results about determinants and linear dependence, which are needed, can be proved directly from the postulates as readily as the manner of stating them in the literature can be modified to fit the case in hand.

The treatment of the algebraic case is the author's. Although it has close connection through the highest common factor with Ritt's excellent discussion, which is based on the division algorithm, it differs radically in several respects from that work because of a difference in purpose and viewpoint. In the first place, the basis of our method is algebra, rather than analysis. Secondly, reducibility, which plays such a prominent rôle in Ritt's developments, is of little importance in ours. With existence theorems as our chief objective, the important thing for us is to eliminate multiple roots. A polynomial's having two factors, for example, does not prevent the application of the implicit function theorem, if the factors are distinct, and making that theorem applicable is the chief purpose of the reduction process. Incidentally, it might be well to point out that the term "reducible" has slightly different meanings in the two theories. The system y^2, which Ritt classes as irreducible, is reducible in ours.

Another feature of our treatment, which assumes its most elegant and satisfactory form in the algebraic case, although employed in the whole work, is the admission of the inequation on an equal footing with the equation. This, together with the use of resultants of all orders (subresultants), obviates the necessity of making the preliminary linear transformation of the indeterminates, which is an essential step in Kronecker's method of solution of algebraic systems.

Finally, the algebraic case furnishes the model for treating the elimination problem for systems of functions. This is done in Chapter VIII. The method is subject to certain limitations. First, there is no algorithm for determining the zeros of an analytic function in a given region. The difficulty of removing this restriction can be appreciated if the zeros of the Riemann ζ-function are cited. Second, there may exist zeros which are not the centers of regions where assumption W is true. These zeros may be termed singular. Their determination and study seem destined to remain for some time a highly complex problem, only to be solved in special cases by special methods. In this respect they resemble the solutions of a system of partial differential equations in the neighborhood of a singular point. In spite of these limitations, the general method of elimination given here seems to furnish a definite result, which is perhaps as satisfactory as can be obtained at present.

In addition to bringing Cartan's existence theorem for pfaffian systems into the scheme, Chapter IX shows clearly that it has limitations because it does not give the singular integral varieties unless substantially modified. The same chapter also gives what is believed to be the only method yet developed for finding and making a partial classification of the singular integral varieties. The method ultimately—and it seems essentially—depends on Riquier's fundamental researches.

In order not to interrupt the continuity of the development, the illustrative examples have been segregated in Chapter XI. The reader may find it convenient to study them at the appropriate place in the text.

The author has drawn freely from the work of Cartan, Goursat, and Janet, but he is particularly indebted to Riquier's treatise. The book also incorporates many suggestions made by students in his courses during the past nine years; the present neat statement of the rule of signs in Theorem 9.1, for example, was suggested by Mr. Alexander Makarov. The author is even more indebted to all those who have listened to his lectures for sustaining his interest in the subject by their sympathetic attention.

J. M. Thomas

July, 1936

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CHAPTER I

INTRODUCTION

The developments in this book are founded upon two types of algebra which we shall in general regard as having a purely formal nature. Each of them is concerned with a set of given symbols, which it combines by four processes called addition, multiplication, identification, and substitution. The following significance, and nothing further, is to be attached to these names. The addition of two symbols A, B in the order indicated means writing them thus: $A + B$. Their multiplication in the order A, B means writing them thus: AB, or when desired, $A \cdot B$. By these two processes compound symbols, such as the AB, for example, are formed.

At the basis of either type of algebra is a set of symbols denoted by $\mathfrak{R}$. Two symbols of $\mathfrak{R}$, such as A and B, which have different appearance are not necessarily distinct. Every symbol of $\mathfrak{R}$ belongs to one and only one of two important subsets of $\mathfrak{R}$ which will be denoted by $\mathfrak{O}$ and $\mathfrak{N}$. All symbols in $\mathfrak{O}$ are to be regarded as identical with the particular symbol 0. Those in $\mathfrak{N}$ are distinct from 0.

The set $\mathfrak{O}$ in particular acquires some of its members when the operations of addition and multiplication are subjected to assumptions, sometimes called laws, which are essentially conventions to the effect that certain compound symbols will be regarded as identical. Identity is denoted by the sign = which will be read "equals" or "is." The assumptions have as logical consequences other statements of identity which they do not formulate explicitly. It is, moreover, often convenient to introduce a new symbol B for a given (compound) symbol A and augment the set of identities $\mathfrak{O}$ by $A - B$. *Identification* is the process of replacing a symbol (in general, occurring in a compound symbol) by another symbol known to be identical with it; applied to a symbol it gives an equal symbol. *Substitution* is the replacement of a symbol by an arbitrarily chosen symbol; it will be applied in particular to the indices on symbols as well as to the symbols themselves. The study of algebra to be made consists in manufacturing compound symbols by the four processes just described and in proving identities among them.

It will be unnecessary to formulate explicitly the assumptions about addition and multiplication of symbols in $\mathfrak{R}$ because that has been done elsewhere in a form which is both elegant and suited to our purpose. We can specify $\mathfrak{R}$ by saying that it is an *integrity domain* [23, I, 39][1] containing an identity symbol with respect to multiplication. Such an $\mathfrak{R}$ can always be imbedded [23, I, 47]

[1] The first number in square brackets refers to the bibliography at the end of the book; the roman number to the volume; and the second arabic number to the page.

in a *commutative field* $\mathfrak{R}^*$, called its *quotient field*. It may happen that $\mathfrak{R} = \mathfrak{R}^*$. This may also be arranged by choosing for $\mathfrak{R}$ a commutative field at the outset.

At times, we shall also regard $\mathfrak{R}$ (or $\mathfrak{R}^*$) as imbedded in another ring $\mathfrak{R}_C$ or $\mathfrak{R}^N$, which in addition to all the properties possessed by $\mathfrak{R}$ have certain others to be specified at the appropriate place. These larger rings are divided into sets $\mathfrak{O}_C$, $\mathfrak{R}_C$, etc.

A symbol y which does not belong to $\mathfrak{R}$ but which behaves in the formal processes of addition and multiplication as if it did will be called an *indeterminate.*

The adjunction of a finite number of indeterminates y to $\mathfrak{R}$ gives a *polynomial ring* $\mathfrak{R}[y_1, \cdots, y_r]$. The algebra of such a ring is the first type of algebra to be considered. The properties of polynomial rings are discussed at length in treatises on modern algebra. Only those results which need to be presented in a special form for our purposes will be developed here and no systematic treatment of the subject will be made.

The adjunction of a finite number of non-commutative *marks* u, which are to be defined later, gives a *Grassmann ring* $\mathfrak{R}[u_1, \cdots, u_n]$, whose algebra constitutes the second type and will be developed systematically from a set of assumptions.

The sum, difference, and product of any two symbols of a ring belong to the ring, which accordingly is said to be closed under addition, subtraction, and multiplication, called the *ring operations.*

CHAPTER II

GENERALITIES ON SYMBOLS AND SYSTEMS

It seems desirable to give in the present chapter certain definitions and theorems in sufficiently general form to answer all our purposes. The chapter can be omitted on a first reading, and the definitions of the terms can be consulted with the aid of the index as they are encountered in subsequent chapters.

1. **Functions of n variables.** Let $y_1, y_2, \cdots, y_n$ be a finite set of symbols, which will be called *variables*. The *scope* of the variables is a set $\mathfrak{A}$ of symbols each of which has the form $(a_1, \cdots, a_n)$, where each a_i belongs to $\mathfrak{R}_C$.

If with the equations

$$y_i = a_i \tag{1.1}$$

is associated the equation

$$f = \text{any symbol of } \mathfrak{B}, \tag{1.2}$$

where the set $\mathfrak{B}$ is determined when a's belonging to $\mathfrak{A}$ are given, and $\mathfrak{B}$ is a subset of $\mathfrak{R}_C$ for all such a's, the symbol f is called a *function* of the variables y. The set $\mathfrak{B}$ is called the *value* of the function.

If there is exactly one symbol on the right of (1.2) for every member of $\mathfrak{A}$, that is, if $\mathfrak{B}$ reduces to a single symbol, so that (1.2) becomes

$$f = b, \tag{1.3}$$

where b is a unique symbol of $\mathfrak{R}_C$, then f is a *single-valued function* of the y's. The word "function" used alone will usually mean "single-valued function."

More generally, if the set $\mathfrak{B}$ contains only a finite number k of symbols, the function is said to have *type* k. Similarly, a set of functions f_i has *type* k if the symbol $(f_1, f_2, \cdots, f_r)$ has associated with it for every $(a_1, a_2, \cdots, a_n)$ from $\mathfrak{A}$ a symbol $(c_1, c_2, \cdots, c_r)$ from a set $\mathfrak{B}$ of k such symbols.

THEOREM 1.1. *If f_i form a set of r functions of $y_1, \cdots, y_n$ whose type is k and g_i form a set of s functions of $y_1, \cdots, y_n$ and $f_1, \cdots, f_r$ whose type is l, then g_i form a set of s functions of $y_1, \cdots, y_n$ whose type is kl.*

The proof consists simply in the remark that the set $\mathfrak{B}$ for g_i as functions of $y_1, \cdots, y_n$ is obtained by combining with an arbitrary member of the $\mathfrak{B}$ for the f's an arbitrary member of that for the g's as functions of both the y's and f's. There are kl such symbols.

The set of all symbols in $\mathfrak{R}_C$ which are not in $\mathfrak{B}$ is called the *complement* of $\mathfrak{B}$ (in $\mathfrak{R}_C$) and is denoted by $\bar{\mathfrak{B}}$. Likewise, the function whose value is $\bar{\mathfrak{B}}$ is called the complement of f (in $\mathfrak{R}_C$) and is denoted by $\bar{f}$.

Formulas (1.1) define a *substitution*, which replaces each y by the a having the same subscript.

The notation $f(y_1, \cdots, y_n)$ for the function f defined above puts in evidence the variables y. Let $f(a_1, \cdots, a_n)$ be used to denote the right member of (1.2). The latter symbol arises from the former by the substitution (1.1).

2. **Systems.** A finite set S of functions each of which has attached to it the name *equation*[2] or *inequation* is called a *system*. Two functions which are both equations or both inequations are said to have the *same nature*.

The inequations are designated by placing bars over them. Thus if S comprises two equations f, g and one inequation h, we write

$$S = f + g + \bar{h}. \tag{2.1}$$

Strictly speaking, we should employ a new symbol rather than the $+$ in (2.1), for S may contain a compound symbol in which the $+$ has already been employed in another sense. We shall avoid this difficulty by enclosing any compound symbol in non-removable parentheses. Thus $S = (f + g)$ will denote a system with the single function $f + g$ and $S = f + g$ will consist of the two functions f, g. More generally, if S and T are two systems, $S + T$ is the system which contains all the equations of S and T as equations and all their inequations as inequations. Likewise, if T is a subsystem of S, then $S - T$ denotes the system obtained by omitting from S the functions of T.

Let a substitution replace the variables in a function f by symbols from their scope. The substitution is called a *zero* or *non-zero* of f according as the result belongs to $\mathfrak{O}_C$ or $\mathfrak{N}_C$.

A substitution is a *root* of S if it is a zero of every equation and a non-zero of every inequation in S. The totality of the roots of S is its *content*. A system S_1 *implies* S_2 and we write $S_1 \geqq S_2$, if every root of S_1 is a root of S_2. Two systems S_1 and S_2 are *equivalent* and we write $S_1 = S_2$, if each implies the other, that is, if they have the same content. If S_1 implies S_2 but is not equivalent to it, we write $S_1 > S_2$.

The system S is said to be *factored* into the two systems S_1, S_2 according to the equation

$$S = S_1 S_2, \tag{2.2}$$

if every root of S is a root of at least one of the factors, and every root of S_1 and every root of S_2 are roots of S. If no sum is involved, it is unnecessary to distinguish between (fg) and fg.

If f is a function, it is clear that $f + \bar{f}$ has no zero and $f\bar{f}$ has no non-zero. Hence we write

$$f + \bar{f} = 1, \qquad f\bar{f} = 0 \tag{2.3}$$

[2] This terminology will save the introduction of additional names, and will lead to no confusion, although "equation" in the ordinary sense means the result of equating the function to zero and not the function itself.

for purposes of manipulation. The equivalences expressed by the following identities are also useful:

$$S^n = S \qquad (n > 1), \tag{2.4}$$

$$S + ST = S, \tag{2.5}$$

$$(S + \bar{f})(S + f) = S. \tag{2.6}$$

In these, S, T represent any systems and f any function.

A system is *consistent* or *inconsistent* according as it has a root or not. In harmony with (2.3) we write $S = 1$, if S is inconsistent. This symbol 1 may be suppressed if it occurs in a product with other factors. It has the further property that $S + 1 = 1$.

A system is inconsistent if it contains a symbol from $\mathfrak{N}$ as equation or a symbol from $\mathfrak{D}$ as inequation.

As will be seen later, a symbol y may have associated with it certain other symbols called its *derivatives*. If each member of a substitution (1.1) is replaced by its derivative of a given type and the result is adjoined to the original substitution, an *extended substitution* results.

If some of the variables are selected and called *unknowns* and the others are interpreted as definite derivatives of those unknowns, a system S becomes a *differential system*. A *solution* of S is a substitution *on the unknowns* which when properly extended becomes a root of S. The definitions of content, equivalence, etc. given above apply to differential systems if the word "root" is replaced by "solution." Thus the *(differential) content* of a differential system is the totality of its solutions, etc.

3. **Ordering symbols.** When clarity will not be impaired, we shall often refer to the symbol $i_1 i_2 \cdots i_n$ as i. The equality $i = j$ will mean

$$i_1 = j_1, \quad i_2 = j_2, \quad \cdots, \quad i_n = j_n. \tag{3.1}$$

Likewise, the inequality $i > j$ (to be read "i is greater than j" or "i follows j") will mean the existence of a positive integer $\lambda \leqq n$ such that

$$i_1 = j_1, \quad \cdots, \quad i_{\lambda-1} = j_{\lambda-1}, \quad i_\lambda > j_\lambda, \tag{3.2}$$

and the inequality $i < j$ (read "i is less than j" or "i precedes j") will mean

$$i_1 = j_1, \quad \cdots, \quad i_{\lambda-1} = j_{\lambda-1}, \quad i_\lambda < j_\lambda. \tag{3.3}$$

If the letters in (3.1), (3.2), and (3.3) represent certain of the rational integers and the signs $=$, $>$, $<$ are given their usual meaning, one and only one of the relations (3.1), (3.2), and (3.3) is verified by any pair i, j. Hence in this case the symbols i, j are said to be *ordered* [23, I, 192].

The above definition can be applied inductively to order complex symbols $\rho = \rho_1 \cdots \rho_m$, where each ρ_α is taken from a previously ordered set $\mathfrak{A}_\alpha$, which may vary with α. In the case that interests us, the symbols $i_1 \cdots i_n$, where each i_λ is a non-negative rational integer, are ordered first; the symbols $\rho_1 \cdots \rho_m$,

where each ρ is a non-negative complex integer $i_1 \cdots i_n$, next; and so on. The ordering is called *lexicographical* because it is used to order the words in dictionaries.

Important properties are given in the following easily proved theorems.

THEOREM 3.1. *Lexicographical ordering is transitive: if $i > j$ and $j > k$, then $i > k$.*

THEOREM 3.2. *Every decreasing sequence of lexicographically ordered symbols is finite.*

THEOREM 3.3. *If $(i_1 \cdots i_m) > (j_1 \cdots j_m)$, then $(i_1 \cdots i_m i_{m+1} \cdots i_n) > (j_1 \cdots j_m j_{m+1} \cdots j_n)$ for arbitrary $i_{m+1}, \cdots, i_n, j_{m+1}, \cdots, j_n$.*

At times, it is better to use the parentheses around the symbol $i_1 \cdots i_n$.

The *sum* of the symbols i, j is defined to be $(i_1 + j_1, \cdots, i_n + j_n)$. Their difference is similarly defined.

4. **Reduction algorithm for systems.** With each symbol of a system S let us associate one of the above symbols $i_r \cdots i_1$, which we shall for convenience temporarily call its *rank* because it becomes the rank in an important special case (§30). Likewise we shall say that a symbol of S has *ordinal*[3] k if its rank is $0 \cdots 0 i_k \cdots i_1$, with i_k, which will be called the *grade*, not equal to zero. Let the symbols of ordinal k form the subset S_k of S.

An operation P_k is called a *reduction algorithm* for systems if it has the following properties:

(i) It is applicable to S so long as S_k contains at least two symbols.

(ii) It leaves unaltered every S_l for $l > k$.

(iii) Each symbol of S_k is omitted or replaced by a symbol not exceeding it in grade. A symbol of ordinal k may be added to S_k provided such symbols added by successive applications of P_k have decreasing rank.

(iv) S_l for $l < k$ is replaced by a finite set of symbols of ordinal l.

(v) There exists a non-negative rational integer a such that P_k^a (i.e., P_k applied a times) replaces at least one symbol of S_k by one with smaller i_k.

(vi) S_k is made to contain at most one inequation by replacing two or more inequations by their product.

We shall next prove: *If $P_r^{c_r} P_{r-1}^{c_{r-1}} \cdots P_1^{c_1}$, where P_k is a reduction algorithm and the c's are appropriately chosen non-negative rational integers, is applied to S, there results a system for which each S_i contains at most one function.*

As P_r is successively applied, the number of symbols in S_r ultimately ceases to increase because by (iii) the additional symbols introduced have decreasing rank and hence by Theorem 3.2 are finite in number. Suppose that, no matter how often P_r is applied, the S_r contains an equation and m other symbols. The grades of these functions form $m + 1$ non-increasing sequences. If only distinct terms are retained, these sequences become decreasing and by Theorem

[3] We do not follow Ritt in attaching the name "class" to this notion because it is necessary to use "class" in a different sense in Chapter IV.

3.2 are finite. Let their minimum members be $d_0, d_1, \cdots, d_m$. There exists an e such that P_r^e gives an S_r with $m + 1$ members having these grades. On the other hand, there exists by (v) an f such that P_r^f decreases at least one d. This contradiction gives the desired result for S_r. The same argument can be successively applied to each of the other S's, and the statement is proved.

We have also proved the useful result contained in

THEOREM 4.1. *A given reduction algorithm can be applied only a finite number of times to a system.*

5. **Ordering by cotes.** The lexicographical ordering has to be modified in order to meet all our needs. Although this modification can be made from a purely abstract viewpoint, we shall develop it in connection with the derivatives

$$\frac{\partial^{i_1+\cdots+i_n} z_\alpha}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \tag{5.1}$$

because the phraseology will be simpler.

The derivatives (5.1) can be given the same order as the complex integers $\alpha i_n \cdots i_1$. In particular, when derivatives of a *single unknown* z are being considered, the symbol $i_n i_{n-1} \cdots i_1$ will be found very useful and will be called the *rank*. This type of ordering will not serve all purposes, however, because a given derivative may have an infinite number of predecessors: thus

$$\frac{\partial z}{\partial x_2} > \frac{\partial^i z}{\partial x_1^i}$$

for all values of i.

The difficulty may be avoided as follows. Let complex integers,[4] called *cotes*,

$$(\gamma_1^\alpha, \cdots, \gamma_s^\alpha), \qquad (1, c_2^j, \cdots, c_s^j) \tag{5.2}$$

be associated with z_α and x_j, respectively, the γ's and c's being non-negative rational integers. The cote of a derivative is defined as the sum of the cotes of the unknown and the independent variables, each of the latter being added as many times as differentiation occurs with respect to that variable. Accordingly, the cote of (5.1) is

$$\begin{aligned}(\gamma_1^\alpha + i_1 + \cdots + i_n, \gamma_2^\alpha + c_2^1 i_1 + \cdots + c_2^n i_n, \cdots, \\ \gamma_s^\alpha + c_s^1 i_1 + \cdots + c_s^n i_n).\end{aligned} \tag{5.3}$$

The derivative with greater cote is defined to be the *follower*. Two different derivatives may, however, have the same cote according to the above

[4] Riquier, who introduced cotes, applied the name to the component rather than to the complex number as we prefer to do.

scheme. Assigning additional cotes to unknowns and independent variables in the following way

$$(5.4)\qquad (\gamma_1^\alpha, \cdots, \gamma_s^\alpha, \alpha, 0, \cdots, 0), \quad (1, c_2^j, \cdots, c_s^j, 0, \delta_n^j, \cdots, \delta_1^j),$$

where δ_k^j is the *Kronecker delta* and is one or zero according as its indices are the same or different, we get the cote

$$(5.5)\qquad (\gamma_1^\alpha + i_1 + \cdots + i_n, \cdots, \gamma_s^\alpha + c_s^1 i_1 + \cdots + c_s^n i_n, \alpha, i_n, \cdots, i_1).$$

Because of Theorem 3.3, if two cotes (5.3) are unequal, the corresponding cotes (5.5) are unequal in the same sense. Hence a change to (5.4) leaves unaltered all the order relations actually established by (5.2). On the other hand, for two derivatives not ordered by (5.2) the first s components of cote (5.5) are equal. If the unknowns z_α and z_β involved are different, the derivatives are ordered as α and β. If the derivatives are of the same unknown, they are ordered by their rank. The new ordering will accordingly be called *complete*.

The following cotes will give a complete ordering:

$$(5.6)\qquad (0, \alpha, 0, \cdots, 0), \qquad (1, 0, \delta_n^j, \cdots, \delta_1^j).$$

By it the derivatives are first separated according to the usual order $(= i_1 + i_2 + \cdots + i_n)$, then according to unknown, and finally according to rank.

If the derivative αi is to precede βj, then

$$\gamma_1^\alpha + i_1 + \cdots + i_n \leqq \gamma_1^\beta + j_1 + \cdots + j_n .$$

Since there is only a finite number of sets of non-negative integral γ_1^α, i_λ satisfying this inequality when its right member is fixed, there is only a finite number of derivatives preceding a given derivative in any ordering by cotes,[5] that is, the derivatives can be written as a sequence D_λ ($\lambda = 1, 2, \cdots$). The integer λ is called the *ordinal* of the derivative. This use of the term will be found in harmony with its previous use. The ordinal reduces to the *order* when $n = r = 1$.

A complete ordering of derivatives by cotes is called *canonical*. We readily prove

THEOREM 5.1. *A canonical ordering has the following properties*:

(i) *it is transitive; that is, if $a > b$ and $b > c$, then $a > c$;*

(ii) *it is invariant under differentiation; that is, if $a > b$ and δ is any differential operator, $\delta a > \delta b$;*

(iii) *if δ is any differential operator, $\delta a > a$;*

(iv) *each derivative has an ordinal as defined above.*

If the variables z, x are subjected to the multiplication

[5] This result may seem strange in the light of the remark that the set of *all* symbols $\lambda_1 \cdots \lambda_m$ does not have the property. The difficulty disappears when one reflects that to an arbitrary $\lambda_1 \cdots \lambda_m$ there does not necessarily correspond a derivative.

$$z_\alpha^* = \zeta_\alpha z_\alpha, \qquad x_i^* = \xi_i x_i, \tag{5.7}$$

where ζ, ξ are constants and repeated indices are not summed, the derivative (5.1), which we shall denote by D, undergoes the multiplication

$$D = \zeta_\alpha \xi_1^{i_1} \cdots \xi_n^{i_n} D^*. \tag{5.8}$$

The coefficient in (5.8) will be called the *ξ-monomial* of the derivative in question. If we put

$$\zeta_\alpha = \theta_1^{\gamma_1^\alpha} \cdots \theta_s^{\gamma_s^\alpha}, \qquad \xi_i = \theta_1 \theta_2^{c_2^i} \cdots \theta_s^{c_s^i}, \tag{5.9}$$

the ξ-monomial becomes a *monomial in the θ's whose exponents are the components* (5.3) *of the cote of the derivative.* The ξ-monomials are given the same ordering as the corresponding derivatives.

A property of canonical ordering which will be useful later is contained in

THEOREM 5.2. *Given a finite set of derivatives in canonical order and a positive ϵ, it is possible to find a multiplication* (5.7) *with coefficients greater than unity such that the ratio of every ξ-monomial to any ξ-monomial following it is less than ϵ.*

From (5.9) it is clear that the theorem is true provided there exist θ's greater than unity such that the θ-monomials have the property in question.

In the case ($s = 1$) of a single θ, the θ-monomials are a finite set of powers of θ arranged according to increasing exponent. The theorem is seen to be true by taking $\theta = 1 + \epsilon^{-1}$.

In the case of s variables, let c be the set of monomials in $s - 1$ variables obtained by making $\theta_1 = 1$ in the given set. The theorem, assumed true for $s - 1$ variables, states the existence of values of $\theta_2, \theta_3, \cdots, \theta_s$, greater than unity, such that the ratio of any monomial of the set c to a monomial following it in c is less than ϵ. For such values of $\theta_2, \theta_3, \cdots, \theta_s$, the assertion of the theorem is seen to be true as applied to two monomials of the original set in which the exponents of θ_1 are the same, provided $\theta_1 > 1$. It remains to choose θ_1 so that the ratio of any monomial $\theta_1^\lambda f$ to any monomial $\theta_1^\mu g$, where $\lambda > \mu$ and f, g do not involve θ_1, is greater than ϵ^{-1}. The conditions are finite in number and are all of the form

$$\theta_1^{\lambda-\mu} > \frac{g}{\epsilon f}.$$

A value of θ_1 greater than unity can clearly be found to satisfy all of them. The theorem is therefore proved in general.

CHAPTER III

GRASSMANN ALGEBRA

In this chapter we shall develop systematically and in modern terminology the algebra invented by Grassmann and so successfully employed by Cartan. It is the second mentioned in the Introduction.

6. **The fundamental ring.** Let $\mathfrak{R}$ be as in Chapter I. To its elements are adjoined n marks $u^1, \cdots, u^n$. Concerning the application of addition and multiplication to the whole set of symbols we make the additional assumptions necessary to insure the truth of the following:

A_1. *For the combination of any three symbols a, b, c the associative and distributive laws are universally valid; that is,*

$$(a + b) + c = a + (b + c), \qquad (ab)c = a(bc),$$
$$(a + b)c = ac + bc, \qquad c(a + b) = ca + cb.$$

In writing sums and products of three or more symbols the parentheses may accordingly be omitted entirely; that is, $a + b + c$ is written for either of the members of the first equation above, etc.

A_2. *The addition of any two symbols a, b is commutative; that is,*

$$a + b = b + a.$$

A_3. *The multiplication of two symbols of $\mathfrak{R}$ or of a symbol of $\mathfrak{R}$ and a mark is commutative; that is,*

$$ab = ba, \qquad au = ua,$$

where a, b belong to $\mathfrak{R}$ and u is a mark.

A_4. *The multiplication of two distinct or identical marks is non-commutative and obeys the law*

$$u^i u^j = -u^j u^i. \tag{6.1}$$

In order that A_4 be self-consistent, the symbols 1 and -1 must be distinct; i.e., the symbols 0, 1 must not form a ring by themselves.[6] That assumption therefore places a restriction on $\mathfrak{R}$ as well as on $\mathfrak{R}[u]$.

[6] The symbol 1 is the multiplication identity in $\mathfrak{R}$ and is not necessarily the integer usually so written. Similarly, $2 = 1 + 1$, etc. In addition to the set α of identities like $2 \cdot 3 = 6$ satisfied by the integers, the symbols $1, 2, 3, \cdots$ may satisfy identities β like $2 = 5$. The use of these same symbols for exponents and indices is justified provided identification is applied to the former only with respect to identities α and is never applied to the latter.

The product

$$m = u^{i_1} \cdots u^{i_p}, \tag{6.2}$$

in which the superscripts are the same as certain of the numbers $1, \cdots, n$, is one of the simplest possible compound symbols. It will be called a *Grassmann monomial*, but since all the monomials in Chapters III and IV are of this type, they will as a rule be called simply monomials without danger of confusion. A given rearrangement of the marks can be effected by permuting the indices $1, 2, \cdots, p$. An even permutation, being equivalent [1, 53] to an even number of transpositions, if applied to (6.2) gives an identical monomial because of A_1 and A_4; an odd gives a result identical with $-m$. The process of performing a given permutation on the indices in (6.2) and multiplying by ± 1 according as the permutation is even or odd will be called *applying the signed permutation*, and the fact just noted is stated as

THEOREM 6.1. *Any Grassmann monomial is invariant under any signed permutation of its indices.*

If two i's in (6.2) are the same symbol, the result of interchanging them is equal to $-m$ by the above and is also obviously equal to m. Hence we have $m = -m$, $2m = 0$. If $\mathfrak{R}$ is a field, multiplication[7] by $\frac{1}{2}$ gives $m = 0$. Accordingly, we put $m = 0$ even when $\mathfrak{R}$ does not contain the reciprocal of 2. This is incorporated in the final assumption that we make for the present:

A_5. *The product* (6.2) *is zero if and only if two of the marks in it are the same. The conditions $ab = 0$, $b \neq 0$ imply $a = 0$ if a belongs to $\mathfrak{R}$.*

From this we have in particular the important inequation

$$u^1 \cdots u^n \neq 0, \tag{6.3}$$

which states a part of the assumption very effectively.

Assumption A_5 can be used to prove $1\, u^i = u^i$, where 1 is the multiplication identity in $\mathfrak{R}$.

It is easily seen that all symbols arising by a finite number of additions and multiplications applied to an $\mathfrak{R}$ and u's which are given and satisfy assumptions A_1 to A_5 inclusive, which will be denoted collectively by A, constitute a special polynomial ring [23, I, 49]. We shall call it a *Grassmann ring* and denote it by $\mathfrak{R}[u^1, \cdots, u^n]$ or, when there is no danger of confusion, by $\mathfrak{R}[u]$. The u's will be called a *basis* of the ring.

A non-zero monomial (6.2) will be called a *unit monomial.*

If two of the marks in a monomial are identical, it will be called *trivial.*

The *grade* of a non-trivial monomial is the number of marks in it. Its *degree* is equal to its grade if the coefficient belongs to $\mathfrak{R}$ or if the monomial (or a polynomial containing it) belongs to a system of which the coefficient is an inequation. The monomial 0 has neither grade nor degree.

[7] Here also the exclusion of the ring 0, 1, for example, becomes necessary.

Since a monomial of grade exceeding n contains at least one repeated mark, every such monomial is zero, and we have

THEOREM 6.2. *The grade of a polynomial in* $\mathfrak{R}[u^1, \cdots, u^n]$ *cannot exceed* n.

The ring is said to have *dimension* n.

7. **Standard form.** A trivial monomial will always be suppressed if it occurs in a sum with other monomials, and replaced by the *standard form* 0, if it stands alone. A properly chosen signed permutation applied to any non-trivial monomial gives rise to a unique equal monomial whose indices are written in increasing order. When this has been done and the coefficient (from $\mathfrak{R}$) written in the leading position, the monomial will be said to be in *standard form.*

Two non-trivial monomials are called *similar* or *dissimilar* according as they contain exactly the same set of marks or not.

The unit monomials (6.2) can be put in a definite *standard order,* say, by the use of cotes (§5).

A polynomial is in *standard form* when all of its monomials have been put in standard form, similar monomials united by addition into a single monomial, and the dissimilar monomials finally written in standard order. This form is unique except for the possibility of adding unit monomials multiplied by zeros.

THEOREM 7.1. *A polynomial all of whose terms are dissimilar is zero if and only if its coefficients are zero.*

Only the necessity of the condition requires proof. This will be accomplished by induction. Let am, where a belongs to $\mathfrak{R}$ and m is unit, be a monomial. By assumption A_5, $am = 0$ implies $a = 0$, and the theorem is true for any monomial.

Now let the polynomial F contain $k + 1$ dissimilar terms, among which are am and $a'm'$, the a's belonging to $\mathfrak{R}$ and the m's being unit. There exists a mark u contained in one of them and not in the other; if this were not so, the monomials would be similar. Suppose, therefore, that $mu \neq 0$ and $m'u = 0$. The polynomial uF has dissimilar terms whose number does not exceed k and among which is aum. Assuming the theorem for k monomials gives $a = 0$. Removal of the corresponding term from F leaves only k dissimilar terms, whose coefficients must therefore be zero by hypothesis. Hence every coefficient is zero and the induction is complete.

THEOREM 7.2. *A polynomial is zero if and only if the coefficients of its standard form are zero.*

If a given unit monomial does not appear in a polynomial, it may be interpreted as being present with zero coefficient. Since two polynomials are equal if and only if their difference is zero, we have

THEOREM 7.3. *Two polynomials are equal if and only if the coefficient of every unit monomial in one is equal to its coefficient in the other.*

The foregoing condition will be abbreviated into: their standard forms have equal coefficients, or into: their standard forms are the same.

8. **Forms.** From many standpoints *forms* are the most important type of polynomial. When the familiar summation convention for repeated indices [7, 3] is employed, the analytic expression for the general form is

$$F = a_{i_1 \cdots i_p} u^{i_1} \cdots u^{i_p}. \tag{8.1}$$

The right member of (8.1) has the appearance of a monomial, whose coefficient will be called a *literal coefficient* of F. In accordance with §6, the integer p is the *grade* of F. The *degree* of F is p if one of its coefficients a is known to belong to $\mathfrak{N}$ or if F forms part of a system containing the inequation a.

It is important to consider the effect of applying a permutation to the indices in the right member of (8.1). If a form equal to F results, F is said to be *invariant under the permutation.* The permutation may be applied to the coefficient alone, to the marks alone, or to both. As a consequence of Theorem 6.1 and a peculiarity of summed indices we have

THEOREM 8.1. *A form is invariant under any permutation applied simultaneously to coefficient and marks. It is invariant under any signed permutation applied in any of the three possible ways.*

As an example we have

$$a_{ijk} u^i u^j u^k = -a_{jik} u^i u^j u^k = +a_{jki} u^j u^k u^i.$$

In accordance with the above, if the coefficient be subjected to all the signed permutations of the symmetric group [1, 54], the $p!$ results are all equal. Hence F can be written as their sum divided by $p!$:

$$F = \frac{1}{p!}(a_{i_1 \cdots i_p} - \cdots) u^{i_1} \cdots u^{i_p}. \tag{8.2}$$

The coefficient in (8.2) has the property that *interchanging any two of its indices is equivalent to multiplying it by* -1. For the application of a transposition to the set of even permutations gives the odd and vice versa [1, 54]. The literal coefficient in (8.2) is therefore called *skew-symmetric*, and the process of passing from (8.1) to (8.2) is described as *rendering the coefficient skew-symmetric.*

Example: $a_{ij} u^i u^j = \frac{1}{2}(a_{ij} u^i u^j - a_{ji} u^i u^j) = \frac{1}{2}(a_{ij} - a_{ji}) u^i u^j$.

The following result is immediate:

THEOREM 8.2. *If the literal coefficient of F is skew-symmetric, the permutations considered in Theorem* 8.1 *leave the monomials of F individually invariant.*

Let us consider the reduction of F, with skew-symmetric literal coefficient, to standard form. Let $i_1, \cdots, i_p$ be an increasing sequence. All monomials in F which contain the corresponding marks are equal by Theorem 8.2 because there is a permutation which if performed on both coefficient and marks carries any one of the monomials into any other. Hence in the standard form $u_1^{i_1} \cdots u_p^{i_p}$

has $p!a_{i_1\cdots i_p}$ for coefficient. If $\bar{a}$ is the skew-symmetric coefficient of an equal form, Theorem 7.3 shows that

$$a_{i_1\cdots i_p} = \bar{a}_{i_1\cdots i_p}. \tag{8.3}$$

These equations, proved on the assumption that the indices are in the natural order, are readily seen to hold for all values of the indices. The result just deduced is expressed by

THEOREM 8.3. *Two forms are equal if and only if their skew-symmetric coefficients are equal.*

The following remarks, although not essential to subsequent developments, in some cases shorten the process of finding the coefficient in (8.2).

Any permutation of the subscripts will be represented by the corresponding permutation of the numbers $1, 2, \cdots, p$. If the signed transposition (km) leaves the coefficient invariant, the coefficient is said to be skew-symmetric in the pair of indices k, m. Let the set S_1 comprise 1 and every k such that the signed transposition $(1k)$ leaves a invariant. Because of the identity $(km) = (1k)(1m)(1k)$, the coefficient is skew-symmetric in every pair of indices from the set S_1. Continuation gives a partition of the symbols $1, 2, \cdots, p$ into sets $S_1, \cdots, S_q$ comprising $p_1, \cdots, p_q$ numbers, respectively. A signed permutation which permutes the indices of every S among themselves leaves a invariant. Hence in rendering a skew-symmetric we need only apply a set of permutations which replaces $S_1, \cdots, S_q$ by every distinct partition of like character. The total number of such partitions is

$$\frac{p!}{p_1! \cdots p_q!},$$

two partitions consisting of the same sets written in different orders being regarded as distinct.

Example: If the literal coefficient is $a_{ij}b_{km}$ with a and b separately skew-symmetric, the six partitions are

$$ij, km; \quad ik, jm; \quad im, jk; \quad jk, im; \quad jm, ik; \quad km, ij.$$

The set of all permutations leaving every S_i invariant is a subgroup of the symmetric group. Let it be used as the first line for tabulating [1, 34] the symmetric group, products being read left to right and the symbol from the first line always appearing on the left. A set of permutations rendering the coefficient skew-symmetric is obtained by selecting an arbitrary permutation, say the first, on each line.

9. **Products of forms.** If $F_1, \cdots, F_m$ are forms, the product $F_1 \cdots F_m$, if it is not zero, is a form whose degree is the sum of the degrees of its factors. It is convenient to have a technique for changing the relative positions of its factors.

Consider first the product of two monomials f, g of grades p, q respectively.

In fg the first mark of g can be put in the leading position by interchanging it successively with each of the p marks of f, proceeding from right to left. Subsequently, the other marks of g can be passed over those of f. Hence

$$fg = (-1)^{pq} gf. \tag{9.1}$$

Next consider the product fgh, where h is a third monomial of degree r. The transposition identity

$$(31) = (23)(21)(23),$$

already employed in §8, enables us to compare fgh and hgf by means of the result just established, and we find

$$fgh = (-1)^s hgf, \tag{9.2}$$

where

$$s = qr + rp + pq. \tag{9.3}$$

From the last expression it is clear that *at least two of the numbers* p, q, r *have the same parity*[8] *as* s. This statement completely determines the parity of s.

Finally, consider the product FGH of three forms. Any monomial in FGH is of the type fgh, where f, g, h are monomials from F, G, H, respectively. The monomials of FGH and HGF are in one-to-one correspondence, with fgh and hgf corresponding. Since corresponding monomials are related by (9.2) and the s is the same for all of them, the forms FGH and HGF are also in the relation (9.2). The presence of additional factors to the right or to the left of FGH obviously does not affect the result, and we have a means of effecting any permutation of the factors in

THEOREM 9.1. *Interchanging two factors of even grade leaves a product of forms invariant; interchanging two factors of odd grade multiplies it by* -1; *interchanging a factor of even grade and one of odd grade multiplies it by* $(-1)^q$, *where* q *is the total grade of the intervening factors.*

THEOREM 9.2. *Any power of a form of odd grade is zero if the exponent exceeds unity.*

10. **Differentiation.** If a non-trivial monomial m contains a mark u, the result of bringing u into the leading position and then making $u = 1$ is called the *derivative* of m with respect to u. If m does not contain u, or if it is trivial, its derivative is found by multiplying it by zero. The derivative of a polynomial, moreover, is defined as the sum of the derivatives of its monomials. The usual symbol for differentiation

$$\frac{\partial^k}{\partial u^{i_1} \cdots \partial u^{i_k}} \tag{10.1}$$

[8] Two numbers have the same parity if both are even or if both are odd; opposite parity, if one is even and the other is odd.

is employed, but attention must be given to the order in which the separate differentiations are performed. In (10.1) it will be understood that the differentiation with respect to u^{i_1} is performed first.

THEOREM 10.1. *The ring $\mathfrak{R}[u]$ is closed under differentiation; that is, if a polynomial belongs to $\mathfrak{R}[u]$, so do all its derivatives.*

To the symbol (10.1) corresponds the monomial

$$u^{i_1} \cdots u^{i_k}. \tag{10.2}$$

If the application of (10.1) to a non-trivial monomial m is to give a non-zero result, m must contain the marks (10.2). Let m be written with (10.2) leading. The derivative (10.1) of m is then found by simply striking out (10.2). Similar considerations hold for any other order of differentiation, and the two differential symbols are in the same relation as the corresponding monomials (10.2). Hence we have

THEOREM 10.2. *The differentiation symbol* (10.1) *is invariant under signed permutation of its indices.*

The processes of differentiation are concisely summarized in

THEOREM 10.3. *In differentiating forms of $\mathfrak{R}[u]$ the rules of the ordinary calculus can be applied provided before each differentiation the factor to be differentiated is placed in the leading position by means of Theorem* 9.1.

Thus if $F = GH$, where G is linear and H cubic,

$$\frac{\partial F}{\partial u} = \frac{\partial G}{\partial u} H - \frac{\partial H}{\partial u} G.$$

Differentiation of (8.1) accordingly gives

$$\frac{\partial F}{\partial u^\alpha} = a_{\alpha i_2 \cdots i_p} u^{i_2} \cdots u^{i_p} - a_{i_1 \alpha \cdots i_p} u^{i_1} u^{i_3} \cdots u^{i_p} + \cdots .$$

Now if the coefficient of F is skew-symmetric, all the terms on the right are equal, and

$$\frac{\partial F}{\partial u^{i_1}} = p a_{i_1 i_2 \cdots i_p} u^{i_2} \cdots u^{i_p}, \tag{10.3}$$

a result which could be written down directly by invoking Theorem 8.2. Repetition gives

$$\frac{\partial^k F}{\partial u^{i_1} \cdots \partial u^{i_k}} = p(p-1) \cdots (p-k+1) a_{i_1 \cdots i_k i_{k+1} \cdots i_p} u^{i_{k+1}} \cdots u^{i_p}, \tag{10.4}$$

whence the formula

$$u^{i_1} \cdots u^{i_k} \frac{\partial^k F}{\partial u^{i_1} \cdots \partial u^{i_k}} = p(p-1) \cdots (p-k+1) F, \tag{10.5}$$

which is the analogue of Euler's theorem. Formulas (10.4) and (10.5) give in particular

$$\frac{\partial^p F}{\partial u^{i_1} \cdots \partial u^{i_p}} = p!\, a_{i_1 \cdots i_p}, \tag{10.6}$$

$$u^{i_1} \cdots u^{i_{p-1}} \frac{\partial^{p-1} F}{\partial u^{i_1} \cdots \partial u^{i_{p-1}}} = p!F. \tag{10.7}$$

11. **Sets of linear forms.** The formulas

$$v^\alpha = a_i^\alpha u^i \qquad (\alpha = 1, 2, \cdots; i = 1, 2, \cdots, n) \tag{11.1}$$

define a set of linear forms. The a's constitute a *matrix* $\| a \|$ of which they are the elements. The matrix is often written as a rectangular array,[9] α denoting the row (horizontal) and i the column (vertical) in which the element a_i^α appears.

Multiplication gives

$$v^{\alpha_1} \cdots v^{\alpha_k} = a_{i_1}^{\alpha_1} \cdots a_{i_k}^{\alpha_k} u^{i_1} \cdots u^{i_k}. \tag{11.2}$$

The result of rendering the coefficient skew-symmetric (in the subscripts, of course) will be denoted by

$$a_{i_1 \cdots i_k}^{\alpha_1 \cdots \alpha_k}/k!,$$

so that

$$k!\, v^{\alpha_1} \cdots v^{\alpha_k} = a_{i_1 \cdots i_k}^{\alpha_1 \cdots \alpha_k} u^{i_1} \cdots u^{i_k}. \tag{11.3}$$

Since the multiplication of the a's in (11.2) is commutative, interchanging the i's on two of them is equivalent to interchanging the corresponding α's. Hence, *the coefficient in* (11.3) *is skew-symmetric in the α's as well as in the i's.*

If all the α's are fixed and distinct and if the same is true of the i's, the a in (11.3) is called the *determinant* of order k comprising the elements in which the rows numbered $\alpha_1, \cdots, \alpha_k$ intersect the columns $i_1, \cdots, i_k$ in the matrix $\| a \|$. The determinant is often represented by placing the square array between single vertical bars $| a |$.

Differentiation of (11.3) with respect to u^{i_1}, substitution of the value of $\partial v/\partial u$ computed from (11.1) and division by k give

$$(k-1)!\,(a_{i_1}^{\alpha_1} v^{\alpha_2} \cdots v^{\alpha_k} - a_{i_1}^{\alpha_2} v^{\alpha_1} \cdots v^{\alpha_k} + \cdots) = a_{i_1 i_2 \cdots i_k}^{\alpha_1 \alpha_2 \cdots \alpha_k} u^{i_2} \cdots u^{i_k}, \tag{11.4}$$

where the terms on the left arise by performing on the α's signed permutations bringing each index successively into the leading position. An equivalent way of writing the left member is

$$a_{i_1}^{\alpha_1} v^{\alpha_2} \cdots v^{\alpha_k} + (-1)^{k-1} a_{i_1}^{\alpha_2} v^{\alpha_3} \cdots v^{\alpha_1} + \cdots, \tag{11.5}$$

[9] This geometrical arrangement of the elements of a matrix or determinant is often very helpful, particularly in numerical calculations. The symbolic methods of the text are not intended to supplant completely the many useful methods suggested by the rectangular array.

where the terms arise by signed cyclic permutation of the subscripts on the α's.[10] Formulas like (11.3) can be used to eliminate from (11.4) the products of $k - 1$ v's appearing on the left:

$$(a_{i_1}^{\alpha_1} a_{i_2 \cdots i_k}^{\alpha_2 \cdots \alpha_k} - a_{i_1}^{\alpha_2} a_{i_2 \cdots i_k}^{\alpha_1 \cdots \alpha_k} + \cdots) u^{i_2} \cdots u^{i_k} = a_{i_1 i_2 \cdots i_k}^{\alpha_1 \alpha_2 \cdots \alpha_k} u^{i_2} \cdots u^{i_k}.$$

The remarks at the end of §8 show that the coefficient on the left is skew-symmetric. Hence by Theorem 8.3 the coefficients on the two sides can be equated to give the identity

$$a_{i_1 \cdots i_k}^{\alpha_1 \cdots \alpha_k} = a_{i_1}^{\alpha_1} a_{i_2 \cdots i_k}^{\alpha_2 \cdots \alpha_k} - a_{i_1}^{\alpha_2} a_{i_2 \cdots i_k}^{\alpha_1 \cdots \alpha_k} + \cdots, \tag{11.6}$$

which is of fundamental importance in determinant theory. The determinant multiplying a given a_i^α on the right, taken with the sign attached to the term, is called the *algebraic complement* of that symbol in the determinant on the left.

The process of deducing (11.6) is interesting in that operations performed in $\mathfrak{R}[u]$ lead to identities among the quantities in $\mathfrak{R}$.

Repetition of the process will give other formulas, which in their totality constitute Laplace's method of expansion.

If every product of $r + 1$ v's is zero and at least one product of r v's is not zero, the set of forms and its matrix are said to have *rank* r.

Formulas (11.6) become for $k = r + 1$, where r is the rank,

$$a_{i_1}^{\alpha_1} a_{i_2 \cdots i_{r+1}}^{\alpha_2 \cdots \alpha_{r+1}} - a_{i_1}^{\alpha_2} a_{i_2 \cdots i_{r+1}}^{\alpha_1 \cdots \alpha_{r+1}} + \cdots = 0.$$

Multiplication by u^{i_1} and use of (11.1) give

$$v^{\alpha_1} a_{i_2 \cdots i_{r+1}}^{\alpha_2 \cdots \alpha_{r+1}} - v^{\alpha_2} a_{i_2 \cdots i_{r+1}}^{\alpha_1 \cdots \alpha_{r+1}} + \cdots + (-1)^r v^{\alpha_{r+1}} a_{i_2 \cdots i_{r+1}}^{\alpha_1 \cdots \alpha_r} = 0.$$

This result will be easier to handle if rewritten with the notation slightly changed as

$$v^{\alpha_1} a_{i_1 \cdots i_r}^{\alpha_2 \cdots \alpha_{r+1}} - v^{\alpha_2} a_{i_1 \cdots i_r}^{\alpha_1 \cdots \alpha_{r+1}} + \cdots + (-1)^r v^{\alpha_{r+1}} a_{i_1 \cdots i_r}^{\alpha_1 \cdots \alpha_r} = 0. \tag{11.7}$$

By hypothesis, for some values of the indices which we shall henceforth consider fixed,

$$a_{i_1 \cdots i_r}^{\alpha_1 \cdots \alpha_r} \neq 0. \tag{11.8}$$

The reciprocal of (11.8) is in $\mathfrak{R}^*$. Multiplication by it converts (11.7) into *a relation expressing all the forms* (since α_{r+1} remains arbitrary) *in terms of the r forms*

$$v^{\alpha_1}, \cdots, v^{\alpha_r}, \tag{11.9}$$

whose product is seen from (11.3) and (11.8) to be different from zero. The forms (11.9) are said to constitute a *basis* of the whole set because of the property proved. A set of linear forms whose rank equals their number is called *independent*. We have

[10] A cyclic permutation of k letters is equivalent to $k - 1$ transpositions [1, 53] because of the identity $(12 \cdots k) = (12)\,(13) \cdots (1k)$.

THEOREM 11.1. *A given independent subset is a basis for a set of linear forms if and only if its rank equals that of the whole set.*

It is readily seen that (11.9), regarded as abbreviations for certain expressions in the marks u, satisfy assumptions A. Hence those v's can be taken as marks defining a Grassmann ring $\mathfrak{R}^*[v^{\alpha_1}, \cdots, v^{\alpha_r}]$ of dimension r. It is called a *sub-ring* of $\mathfrak{R}^*[u^1, \cdots, u^n]$ because the latter contains all its members.

Consider now the case $r = n$. Since the product of any $n + 1$ forms is zero, the adjunction of the linear forms $u^1, \cdots, u^n$ to the set of v's does not alter the rank. Hence the u's can be expressed in terms of the v's and *every polynomial of* $\mathfrak{R}^*[u]$ *belongs to* $\mathfrak{R}^*[v]$. These two rings are therefore regarded as the same ring referred to different bases. If the notation is adjusted so that (11.9) are the first n v's, the formulas defining the change of basis are

$$v^i = a^i_j u^j \qquad (i, j = 1, 2, \cdots, n). \tag{11.10}$$

Because the u's belong to $\mathfrak{R}^*[v]$ there are also the companion formulas

$$u^j = A^j_k v^k. \tag{11.11}$$

Substitution from (11.11) in (11.10) and equating coefficients give

$$a^i_j A^j_k = \delta^i_k , \tag{11.12}$$

where the right member is called the Kronecker delta (§5). The matrices $\| a \|$ and $\| A \|$ are called *inverses*. By remarking that the relations between the two sets of marks is reciprocal or by substitution from (11.10) in (11.11) we get the companion formulas

$$A^i_j a^j_k = \delta^i_k , \tag{11.13}$$

which are equivalent to (11.12).

In (11.7) put $\alpha_1 \cdots \alpha_r = i_1 \cdots i_r = 1 \cdots n$, $a^{1 \cdots n}_{1 \cdots n} = a$, and $v^{\alpha_{r+1}} = u^i$ so that $a^{\alpha_{r+1}}_j = \delta^i_j$, and get a result of the form

$$au^i = \mathfrak{A}^i_j v^j. \tag{11.14}$$

Substitution from (11.14) in (11.10) multiplied by a gives the first, and substitution from (11.10) in (11.14) gives the second, of the relations

$$a^i_j \mathfrak{A}^j_k = \mathfrak{A}^i_j a^j_k = \delta^i_k a. \tag{11.15}$$

From these and the definition [cf. (11.6)], or from (11.7), $\mathfrak{A}^j_k$ is readily identified as the algebraic complement of a^k_j in a. The advantage of (11.14), (11.15) over (11.11), (11.12), (11.13) is that the coefficients $\mathfrak{A}$ all belong to $\mathfrak{R}$, whereas the A's are only known to belong to $\mathfrak{R}^*$, that is, the A's are rational in the a's, whereas the $\mathfrak{A}$'s are rational and integral. The connection between the A's and $\mathfrak{A}$'s is furnished by the formulas

$$\mathfrak{A}^i_j = aA^i_j . \tag{11.16}$$

The condition for the existence of A's satisfying (11.12) in which the a's are given is of course that the v's be the marks of a ring of dimension n; that is, the condition like (6.3) must be satisfied. Since

(11.17) $$v^1 \cdots v^n = au^1 \cdots u^n,$$

where a is the determinant of all the a_j^i, the condition is

(11.18) $$a \neq 0.$$

An interesting by-product of the above is the multiplication theorem for determinants. Employing the transformation of marks $u^j = b_k^j t^k$ along with (11.10), we find

$$v^i = a_j^i b_k^j t^k, \qquad v^1 \cdots v^n = abt^1 \cdots t^n,$$

where b is the determinant with elements b. Comparison of these formulas with (11.10) and (11.17) shows that ab is the determinant with elements

(11.19) $$a_j^i b_k^j .$$

In terms of the marks v, form (8.1), which we assume to have skew-symmetric coefficients, becomes

$$a_{i_1 \cdots i_p} A_{j_1}^{i_1} \cdots A_{j_p}^{i_p} v^{j_1} \cdots v^{j_p},$$

that is, a form with coefficient

(11.20) $$b_{j_1 \cdots j_p} = a_{i_1 \cdots i_p} A_{j_1}^{i_1} \cdots A_{j_p}^{i_p} .$$

If all the a's are zero, so are the b's. Consequently, equality of polynomials is independent of the basis. In particular, let a set of linear forms have rank r and s in $\mathfrak{R}[u]$ and $\mathfrak{R}[v]$, respectively. Since all products of $r + 1$ of the forms are zero in $\mathfrak{R}[u]$, they are zero in $\mathfrak{R}[v]$ and $s \leqq r$. The same argument shows that $r \leqq s$, so that $r = s$, and we have

THEOREM 11.2. *The rank of a set of linear forms is invariant under change of basis.*

Returning now to the general case, let (11.8) hold and let $i_{r+1}, \cdots, i_n$ be chosen so that $i_1, \cdots, i_n$ is a permutation of $1, \cdots, n$. In accordance with (11.13), there exist in $\mathfrak{R}^*$ B's satisfying

$$B_{\alpha_j}^{i_k} a_{i_l}^{\alpha_j} = \delta_l^k \qquad (j, k, l = 1, \cdots, r).$$

Multiplication of

$$v^{\alpha_j} = a_{i_l}^{\alpha_j} u^{i_l} + a_{i_p}^{\alpha_j} u^{i_p} \qquad (p = r + 1, \cdots, n)$$

by $B_{\alpha_j}^{i_k}$ gives

(11.21) $$B_{\alpha_j}^{i_k} v^{\alpha_j} = u^{i_k} + B_{\alpha_j}^{i_k} a_{i_p}^{\alpha_j} u^{i_p}.$$

The forms on the right clearly have rank r because the monomial

$$u^{i_1} \cdots u^{i_r} \tag{11.22}$$

has coefficient 1 in their product. They therefore constitute for the set of v's a basis which will be called *normalized in* $\mathfrak{R}^*$ *with respect to the marks* (11.22).

The B's in (11.21) are rational functions of the a's. If (11.15) are employed, the formulas

$$\mathfrak{B}^{i_k}_{\alpha_j} v^{\alpha_j} = Du^{i_k} + \mathfrak{B}^{i_k}_{\alpha_j} a^{\alpha_j}_{i_p} u^{i_p}, \tag{11.23}$$

in which the $\mathfrak{B}$'s belong to $\mathfrak{R}$, are obtained.

The forms (11.9) taken with $u^{i_{r+1}}, \cdots, u^{i_n}$ have a non-zero product and hence can be taken as a basis of $\mathfrak{R}^*[u]$. This is stated in

THEOREM 11.3. *A set of linear forms whose rank is r determines a subring of dimension r any basis of which can be made the first r marks in a basis for the whole ring.*

Any non-zero product of forms in the subring $\mathfrak{R}[v]$ is different from zero in $\mathfrak{R}[v^{\alpha_1}, \cdots, v^{\alpha_r}, u^{i_{r+1}}, \cdots, u^{i_n}]$, and any product which is zero in the former is zero in the latter. Hence the rank of a given set of forms belonging to these two rings is the same in both. But the rank in the second ring is the same as in $\mathfrak{R}[u]$ by Theorem 11.2. Hence

THEOREM 11.4. *The rank of a set of linear forms in a subring is the same as its rank in the whole.*

Because of Theorem 11.3, when a theorem whose truth is independent of the basis is to be proved, it may be assumed that any given set of linear forms of rank r is $u^1, \cdots, u^r$. This will be illustrated in the proof of

THEOREM 11.5. *In a set of linear forms v^α let $v^1, \cdots, v^k$ be an arbitrary but fixed subset of rank k. The whole set is of rank r if and only if every product of $r + 1$ forms among which are $v^1, \cdots, v^k$ is zero, whereas some product of r forms among which are $v^1, \cdots, v^k$ is not zero.*

To prove the sufficiency, all products of degree $r + 1$ must be shown to vanish. We may identify $v^1, \cdots, v^k$ with $u^1, \cdots, u^k$. By hypothesis

$$u^1 \cdots u^k v^{\alpha_{k+1}} \cdots v^{\alpha_{r+1}} = 0. \tag{11.24}$$

If we show that this relation persists when any u is replaced by any v, we shall have the desired result.

Differentiation of (11.24) with respect to u^1 gives

$$u^2 \cdots u^k v^{\alpha_{k+1}} \cdots v^{\alpha_{r+1}} \pm u^1 \cdots u^k \frac{\partial v^{\alpha_{k+1}}}{\partial u^1} v^{\alpha_{k+2}} \cdots v^{\alpha_{r+1}} \pm \cdots = 0.$$

Multiplication by v^{α_1} and use of (11.24) gives what is wanted, namely,

$$v^{\alpha_1} u^2 \cdots u^k v^{\alpha_{k+1}} \cdots v^{\alpha_{r+1}} = 0.$$

Hence the condition is sufficient.

The first part of the condition is obviously necessary. On the other hand, if all the products mentioned in the second part were zero, all products of degree r would be zero by the sufficiency proof just given.

For $k = r$, Theorem 11.5 implies the following result from determinant theory: any non-vanishing determinant in a matrix of rank r is contained in at least one non-vanishing determinant of order r. It likewise is closely related to the theorem (which, however, is not an immediate consequence of it): a matrix is of rank r if and only if every determinant of order $r + 1$ which contains a given non-vanishing determinant of order r vanishes.

12. Associates and adjoints. The substitution

$$u^{i_1} = x_1^{i_1}, \quad \cdots, \quad u^{i_p} = x_p^{i_p}, \tag{12.1}$$

in which the x's are indeterminates (see Chapter I), applied to the form (8.1), whose coefficient has been rendered skew-symmetric, gives a unique *skew-symmetric p-linear form*

$$a_{i_1 \cdots i_p} x_1^{i_1} \cdots x_p^{i_p} \tag{12.2}$$

of the polynomial ring $\mathfrak{R}[x]$. The inverse substitution (12.1) applied to any skew-symmetric p-linear form gives a unique form of $\mathfrak{R}[u]$ with skew-symmetric coefficient. Two forms (8.1) and (12.2) related in this manner will be called *associates.*[11]

It now becomes convenient to discuss simultaneously with $\mathfrak{R}[u^1, \cdots, u^n]$ a second ring $\mathfrak{R}[u_1, \cdots, u_n]$, which is of the same dimension and arises from the same $\mathfrak{R}$ as that already considered, but whose marks are different and are denoted by u's with subscripts. When desirable, the two rings will be distinguished by the adjectives *upper* and *lower*. For our purposes, not many properties of the set of $2n$ marks taken as a whole are needed, but it is convenient to make the following assumption which implies all that is necessary.

A_6. $\mathfrak{R}[u^1, \cdots, u^n; u_1, \cdots, u_n]$ *is a Grassmann ring of dimension* $2n$.

Given the ring $\mathfrak{R}[u^1, \cdots, u^n]$, the adjunction of the additional marks to it will always give a ring having the properties of that whose existence is postulated in A_6.

The marks of the two rings can, of course, be subjected to transformations which are completely independent of each other. But if the upper ring is subjected to (11.10) and it is required that $u^i u_i$ be invariant, i.e., that

$$v^i v_i = u^j u_j, \tag{12.3}$$

[11] The fact that there is a one-to-one correspondence between the theorems deduced for forms of $\mathfrak{R}[u]$ and the theorems for skew-symmetric p-linear forms gives us a criterion for the consistency [22, 2] of our assumptions; that is, the theory of forms in $\mathfrak{R}[u]$ is just as consistent as that of p-linear skew-symmetric forms.

the transformation of the lower ring is determined as

$$u_j = a_j^i v_i. \tag{12.4}$$

For substitution from (11.10) in (12.3) gives

$$u^j(u_j - a_j^i v_i) = 0.$$

This equation expresses that a certain quadratic form in the ring of dimension $2n$ is zero. Since no upper marks occur in the parentheses, dissimilar terms in the parentheses lead to dissimilar terms in the whole. Hence by Theorem 7.2 the quantity in parentheses is zero and (12.4) is true.

An inspection of (11.10) and (11.11) shows that formulas (12.4) imply in $\mathfrak{R}^*[u_1, \cdots, u_n]$

$$v_j = A_j^i u_i, \tag{12.5}$$

in which the new marks are on the left as in (11.10). These formulas are also easily deduced from (12.4) by use of (11.12).

Transformations (11.10) and (12.4) are called *contragredient.* To pass from one to the other we raise (or lower) the indices on both marks and then interchange the marks carrying along the attached indices.

Let the form (8.1) be multiplied by $n - p$ linear forms $w^{p+1}, \cdots, w^n$ with indeterminates x for coefficients to give a form of degree n in the upper ring:

$$Fw^{p+1} \cdots w^n = fu^1 \cdots u^n. \tag{12.6}$$

Its coefficient, when rendered skew-symmetric, is a skew-symmetric $(n - p)$-linear form in the indeterminates. The associate of f, obtained by the substitution

$$x_{i_{p+1}}^{p+1} = u_{i_{p+1}}, \quad \cdots, \quad x_{i_n}^n = u_{i_n},$$

is called the *adjoint* of F and is denoted by F^*.

If (11.10) is applied to the marks in the linear form

$$w = x_i u^i,$$

there results a linear form whose coefficients are

$$y_j = A_j^i x_i.$$

Comparison with (12.5) shows that the coefficients undergo the transformation contragredient to (11.10). Consequently any set of indeterminates with a fixed upper index in the associate of the adjoint undergo the contragredient transformation, and the same is true of the marks in the adjoint. If F becomes G, from (12.4) and the analogue of (11.17), namely,

$$u_1 \cdots u_n = av_1 \cdots v_n,$$

the associate of the adjoint is given by $f = ag$. This invariance of the adjoint is expressed in

THEOREM 12.1. *If a transformation of marks converts F into G, the contragredient transformation converts F^* into aG^*, where a is the determinant of the transformation.*

If F and G are any two forms of the same degree and a belongs to F, the following clearly hold:

$$(F + G)^* = F^* + G^*, \qquad (aF)^* = aF^*. \tag{12.7}$$

Finding the adjoint of a form is thereby reduced to finding the adjoint of the monomial

$$u^{i_1} \cdots u^{i_p}. \tag{12.8}$$

The variables (12.8) can be omitted from the forms w, whose product thus becomes

$$w^{p+1} \cdots w^n = xu^{i_{p+1}} \cdots u^{i_n},$$

where x is the determinant of order $(n - p)$

$$x^{p+1}_{i_{p+1}} \cdots x^n_{i_n} - \cdots . \tag{12.9}$$

Since

$$u^{i_1} \cdots u^{i_p} w^{p+1} \cdots w^n = xu^{i_1} \cdots u^{i_n},$$

from (12.9) the adjoint of (12.8) is

$$\pm\, u_{i_{p+1}} \cdots u_{i_n},$$

the sign being that of the permutation $i_1 \cdots i_n$.

Note that in defining adjoint the linear forms were placed at the right. If the definition were modified by placing them on the left, the adjoint would be $(-1)^{p(n-p)}$ times the expression originally defined.

For a form in the lower ring the adjoint is defined by placing the linear forms on the left. From the discussion of the adjoint of a monomial we have

THEOREM 12.2. *The adjoint relationship is reciprocal.*

13. **Generalization of linear dependence.** In §11 it was seen that the forms of a linear set can be expressed as linear combinations of a subset, or as is often said, are *linearly dependent* on the subset. We proceed to generalize this notion of dependence.

Let F be a form of degree p. Because of the non-commutative law of multiplication, no term of F contains a power of u^1 higher than the first. By collecting the terms containing u^1, we may therefore write

$$F = u^1 v_1 + \varphi, \tag{13.1}$$

where neither v_1 nor φ involves u^1. Repetition of this process gives

$$F = u^\alpha v_\alpha \qquad (\alpha = 1, 2, \cdots, n - p + 1), \tag{13.2}$$

where the u's involved in any v have index greater than the v, and in particular $v_{n-p+1} = au^{n-p+2} \cdots u^n$, a belonging to $\mathfrak{R}$. The separation (13.2) may of course be repeated on each v.

Suppose that

$$F = u^1\varphi_1 + \cdots + u^k\varphi_k. \tag{13.3}$$

Multiplication gives

$$u^1 \cdots u^k F = 0. \tag{13.4}$$

Conversely, if we assume (13.4) and use (13.2) in it, we get equations of the type

$$u^1 \cdots u^k u^{k+1} v_{k+1} = 0. \tag{13.5}$$

If v_{k+1} is put in standard form, each of its monomials leads to a monomial in the standard form of the left member of (13.5) with the same coefficient. Hence all the coefficients of v_{k+1} are zero. Since this applies to all v_α with $\alpha > k$, (13.2) has only k terms, and gives a special form of (13.3). We therefore have the useful result first stated by Cartan

THEOREM 13.1. *If $v^1, \cdots, v^k$ are linear forms of rank k, the equations*

$$F = v^1\varphi_1 + \cdots + v^k\varphi_k \tag{13.6}$$

and

$$v^1 \cdots v^k F = 0 \tag{13.7}$$

are equivalent in $\mathfrak{R}^*[u]$.

14. **The associated set.** The $(p - 1)$th derivatives of the form (8.1) are linear. Their totality constitutes the *associated set* of F. Let $w^1, \cdots, w^r$ be a basis for it. The subring $\mathfrak{R}^*[w^1, \cdots, w^r]$ will be called the *subring* of F in $\mathfrak{R}^*[u^1, \cdots, u^n]$, and r, the *rank* of F.

If the basis is changed by (11.10), the new associated set has for members

$$\frac{\partial^{p-1}F}{\partial v^{i_1} \cdots \partial v^{i_{p-1}}} = A^{j_1}_{i_1} \cdots A^{j_{p-1}}_{i_{p-1}} \frac{\partial^{p-1}F}{\partial u^{j_1} \cdots \partial u^{j_{p-1}}}, \tag{14.1}$$

whence they belong to $\mathfrak{R}^*[w^1, \cdots, w^r]$, and their rank $s \leqq r$. On the other hand, formulas similar to (14.1) show that the original associated set belongs to the subring determined by the new and that $r \leqq s$. Hence the two subrings are identical and $r = s$.

The associated set of a finite system of forms F^α, whose degree p_α possibly varies with α, is defined as the union of the associated sets. It determines, as in the case of a single form, the subring and rank of the system, and we clearly have from the above argument

THEOREM 14.1. *The subring of a system of forms in $\mathfrak{R}^*[u]$ and their rank are invariant under change of basis.*

Letting $\mathfrak{R}^*[u^1, \cdots, u^r]$ be the subring of the system of forms F^α and multiplying (10.7) by $u^1 \cdots u^r$ we get

$$u^1 \cdots u^r F^\alpha = 0.$$

Because of Theorem 13.1 formula (13.2) may be assumed as

$$F^\alpha = u^\beta v^\alpha_\beta \qquad (\beta = 1, 2, \cdots, r). \tag{14.2}$$

We wish to prove

THEOREM 14.2. *All the forms of a finite system belong to the subring of that system.*

The proof will be accomplished by induction on the maximum degree of the forms in the system. If the system contains only linear forms, it coincides with its associated set, and the theorem is obviously true for it.

Differentiation of (14.2) with respect to marks numbered $\beta i_1 \cdots i_{p-2}$ and multiplication by $u^1 \cdots u^r$ give

$$u^1 \cdots u^r \frac{\partial^{p-2} v^\alpha_\beta}{\partial u^{i_1} \cdots \partial u^{i_{p-2}}} = 0.$$

Because of Theorem 11.3, these equations state that if $u^1, \cdots, u^r$ be augmented by the associated set of the system v^α_β, the rank of the whole set is r; that is, the subring of the system of v's belongs to $\mathfrak{R}^*[u^1, \cdots, u^r]$. If the theorem is assumed true for maximum degree one less than that of the system F^α, the v's belong to their subring and hence to $\mathfrak{R}^*[u^1, \cdots, u^r]$. From (14.2) it is then apparent that the F's belong to their subring and the induction is complete.

THEOREM 14.3. *The rank of a system of forms is the minimum number of marks in terms of which they can be expressed by a change of basis.*

The above theorem is true for the following reason. If the system were expressed in terms of fewer than r marks, its associated set would involve fewer than r marks and would have rank less than r.

THEOREM 14.4. *A mark u appears in at least one F^α if and only if it appears in at least one form of the associated set.*

To prove the first part, we remark that only the marks in the F^α can appear in their associated set, for differentiation introduces no new marks. To prove the second, we note that since F^α can be expressed in terms of the associated set, the absence of a mark from the associated set would mean its absence from F^α.

If in formulas (11.10) we substitute for the u's the forms of a given degree in a system F^α, n being interpreted as the number of such forms, formulas (11.10) give new forms of the same number and the given degree. Corresponding to each degree represented in the system F^α we get a set of formulas

$$G^\rho = a^\rho_\sigma F^\sigma. \tag{14.3}$$

The set of forms G^α so obtained is said to arise from F^α by a linear transformation of forms. Multiplication of (14.3) by A^τ_ρ and the use of (11.13) give

$$F^\rho = A^\rho_\sigma G^\sigma. \tag{14.4}$$

Since the $(p-1)$th derivatives of a given type undergo the same transformation as the forms, we have

THEOREM 14.5. *The subring of a set of forms and the rank of the set are invariant under linear transformation of the forms.*

15. **Factorization.** The notion of factor, namely, that G and H are factors of F if $F = GH$, is clearly independent of change of basis. The case of linear factors is particularly important. Since by Theorem 13.1 when $u \neq 0$, $F = u\varphi$ and $uF = 0$ are equivalent, we have the following criterion for divisibility by a given linear factor in $\mathfrak{R}^*[u]$:

THEOREM 15.1. *A form F has the linear factor u if and only if $uF = 0$.*

Reference to (13.1) shows that the condition for divisibility by u^1 is $\varphi = 0$, and for simultaneous divisibility by u^2 is

$$u^1 u^2 v_1 = 0.$$

This may be written $u^2 v_1 = 0$. Hence v_1 is divisible by u^2. Continuing this reasoning, we find the two following results:

THEOREM 15.2. *A form divisible by $u^1, \cdots, u^k$ of rank k is divisible by their product.*

THEOREM 15.3. *Any form F can be written*

$$F = u^1 \cdots u^k G, \tag{15.1}$$

where G has no linear factors. A linear form is then a factor of F if and only if it belongs to $\mathfrak{R}^*[u^1, \cdots, u^k]$.

If F has degree $n-1$, its adjoint is linear. Let this linear form be made u_n by transformation of the lower ring. By Theorems 12.2 and 12.1 the contragredient transformation converts F into $au^1 \cdots u^{n-1}$; that is, F is the product of $n-1$ linear factors. This result will be expressed by

THEOREM 15.4. *Every form of degree $n-1$ in a ring of dimension n is monomial.*

A consequence of the above is that the rank of a form cannot be equal to its degree plus unity.

Let

$$\Phi = u^1 \cdots u^n. \tag{15.2}$$

The derivative $\partial\Phi/\partial u^i$ can be obtained from the adjoint of u^i simply by raising all the subscripts to superscripts. It will be called the *adjoint* of u^i, when there is no chance of confusion.

The first derivatives of Φ comprise all products of $n - 1$ u's. Hence any form of degree $n - 1$ can be written

$$F = x^i \frac{\partial \Phi}{\partial u^i}. \tag{15.3}$$

The conditions (Theorem 15.1) that F be divisible by every form of the set (11.1) are found to be

$$a_i^\alpha x^i = 0. \tag{15.4}$$

The left members of these *linear, homogeneous* equations are the associates of the linear forms (11.1). Symbols x^i which are in $\mathfrak{R}$ and are not all zero satisfy (15.4) if and only if they are the coefficients of a form of degree $n - 1$ divisible by every linear form in the ring of the v's. Accordingly, to get a root of (15.4), multiply the product of the forms (11.9) by a form of degree $n - r - 1$. The symbol multiplying the adjoint of u^i in the result is the x^i of the root. If the coefficients of the form used as a multiplier are arbitrary, a *general* root results.

If $r < n$, the above process always gives a root of (15.4) in which some x is different from zero. If $r = n$, however, we cannot multiply by a form of degree $n - r - 1$. The only root is $x^i = 0$, the so-called *trivial* root always present.

THEOREM 15.5. *A linear homogeneous system of equations has a non-trivial root if and only if its rank is less than the number of variables.*

To determine whether a *given* form G is a factor of F leads to a system of non-homogeneous equations. To obtain the system, put $F = GH$, where H has indeterminates for coefficients, and equate coefficients on both sides. A criterion for the existence of a root is contained in the theorem of the following section.

To test whether F is factorable into GH, where only the degrees of G and H are specified, leads to a system of quadratic equations and is much more complicated.

16. **Systems of linear homogeneous equations.** Although the viewpoint of the preceding section leads to an elegant and concise treatment of the theory of these systems, the development of the details is beyond the scope of this work. We find it convenient, however, to prove the following existence theorem which is in a sense complete and therefore more satisfactory than the partial result given in the last section.

THEOREM 16.1. *A system of linear homogeneous equations with coefficients in $\mathfrak{R}$ has a root in $\mathfrak{R}$ with a given set of the variables having arbitrary values in $\mathfrak{R}$ if and only if replacing the corresponding marks by zero leaves the rank of the set of associates of the left members invariant.*

The operation mentioned amounts to annihilating the columns corresponding to the given variables in the matrix.

Let $i_1, \cdots, i_n$ be a permutation of $1, \cdots, n$. Employing the notation of §11 suppose

$$a = a_{i_{n-r+1}\cdots i_n}^{\alpha_{n-r+1}\cdots\alpha_n} \neq 0. \tag{16.1}$$

Define the form which is to give the root by

$$a\Psi = x^\lambda \frac{\partial \Pi}{\partial u^\lambda}, \qquad \Pi = u^{i_1} \cdots u^{i_{n-r}}, \qquad (\lambda = i_1, \cdots, i_{n-r}).$$

Multiplication gives

$$\Psi v^{\alpha_{n-r+1}} \cdots v^{\alpha_n} = x^\lambda \frac{\partial \Phi}{\partial u^\lambda} + \Pi G, \tag{16.2}$$

where G is linear in the x's. Thus Ψ gives a root in which the prescribed set of $n - r$ variables x^λ have arbitrary values. On the other hand, if the determinant (16.1) is zero for all choices of the α's, for an arbitrary Ψ the left member of (16.2) will be divisible by Π and the variables x^λ are zero in every root. Theorem 16.1 is therefore true. The analogy of the manipulation just employed with that used in defining the adjoint is apparent.

A non-homogeneous system is made homogeneous by the introduction of an additional variable, whose value must be 1. It is clear that a homogeneous system has a root in which a given variable is 1 if and only if it has a root in which that variable is arbitrary. Hence Theorem 16.1 gives a criterion for the existence of a root in $\mathfrak{R}^*$ for a system of non-homogeneous linear equations with coefficients in $\mathfrak{R}$.

If the system has rank n and is

$$a_i x^i = b^i, \tag{16.3}$$

its solution can be obtained in the form

$$a x^i = \mathfrak{A}_j^i b^j. \tag{16.4}$$

It is sometimes convenient to denote the result of substituting n symbols b^i for the jth column of a by $a[b, j]$. In terms of this notation (16.4) becomes

$$a x^i = a[b, i].$$

17. **A quadratic form in the presence of linear forms.** Let F be quadratic and $u^1, \cdots, u^h$ linear forms with $u^1 \cdots u^h \neq 0$. There is a least integer k satisfying

$$u^1 \cdots u^h F^{k+1} = 0 \tag{17.1}$$

because the degree of a form in $\mathfrak{R}[u]$ cannot exceed n. Equation (13.2) may be written

$$F = u^1 v_1 + u^2 v_2 + \cdots + u^{h+1} v_{h+1} + \varphi, \tag{17.2}$$

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where φ involves only u's of index exceeding $h + 1$ and any v involves only u's with indices exceeding its own. Multiplication gives

$$(17.3) \qquad u^1 \cdots u^h F^{k+1} = (k + 1)u^1 \cdots u^{h+1} v_{h+1} \varphi^k + u^1 \cdots u^k \varphi^{k+1}.$$

Since the first term contains u^{h+1} and the second does not, they must vanish separately. Consequently

$$(17.4) \qquad v_{h+1} \varphi^k = 0.$$

Replacing k in (17.3) by $(k - 1)$, multiplying by v_{h+1} and using (17.4) gives

$$(17.5) \qquad u^1 \cdots u^h v_{h+1} F^k = 0.$$

If every v with index exceeding h in (17.2) is zero

$$u^1 \cdots u^h F = 0$$

so that $k = 0$. For $k > 0$, therefore, there surely exists a linear form v_{h+1} satisfying (17.5) and

$$(17.6) \qquad u^1 \cdots u^h v_{h+1} \neq 0.$$

Let any linear form satisfying (17.5) and (17.6) be denoted by u^{h+1}, and let l be the least integer satisfying

$$u^1 \cdots u^{h+1} \neq 0, \qquad u^1 \cdots u^{h+1} F^{l+1} = 0.$$

The above process can now be repeated to augment the number of u's and diminish the degree of F in these conditions. Since $l < k$, after at most k steps we have

$$(17.7) \qquad u^1 \cdots u^{h+s} \neq 0, \qquad u^1 \cdots u^{h+s} F = 0, \qquad s \leqq k.$$

Theorem 13.1 gives

$$(17.8) \qquad F = u^\alpha v_\alpha \qquad (\alpha = 1, \cdots, h + s).$$

In the product

$$(17.9) \qquad u^1 \cdots u^h F^k \neq 0$$

every term is of degree $k + h$ in the $s + h$ u's. Hence $s < k$ is untenable, there are exactly $s = k$ steps in obtaining (17.7), and the power of F appearing in the equation like (17.1) diminishes by exactly unity each time. Moreover, an appeal to the form (17.8) resulting from the complete process shows that the non-vanishing form (17.9) at any stage is monomial. It has already been seen that any one of its factors not in $\mathfrak{R}^*[u^1, \cdots, u^h]$ can be used as u^{h+1}.

THEOREM 17.1. *If F is quadratic, $u^1, \cdots, u^h$ linear and*

$$F^k u^1 \cdots u^h \neq 0, \qquad F^{k+1} u^1 \cdots u^h = 0,$$

the form

$$F^k u^1 \cdots u^h$$

is monomial. If u^{h+1} is any one of its factors satisfying $u^1 \cdots u^{h+1} \neq 0$,

$$F^{k-1} u^1 \cdots u^{h+1} \neq 0, \qquad F^k u^1 \cdots u^{h+1} = 0.$$

Differentiation of (17.1) with respect to u^1 gives the useful identity

$$F^{k+1} u^2 \cdots u^h + (k+1) \frac{\partial F}{\partial u^1} F^k u^1 \cdots u^h = 0. \tag{17.10}$$

18. **The canonical form of a quadratic form.** If the set of u's in the preceding section is initially vacuous ($h = 0$), it follows from (17.9) that the $2k$ linear forms in (17.8) are independent. Moreover, the rank r of F satisfies $r \leqq 2k$. The inequality would violate (17.9), so that $r = 2k$, and we have

THEOREM 18.1. *The rank of a quadratic form is double the integer k defined by*

$$F^k \neq 0, \qquad F^{k+1} = 0. \tag{18.1}$$

Equation (17.8) becomes by slightly changing the notation

$$F = U^1 U^2 + \cdots + U^{r-1} U^r \qquad (r = 2k). \tag{18.2}$$

By the process of the preceding section the U's with odd index are obtained successively to satisfy the linear systems

$$F^k U = 0; \qquad F^{k-1} U^1 U = 0; \qquad F^{k-2} U^1 U^3 U = 0; \qquad \cdots.$$

The right member of (18.2) is called the *canonical form* of F. The U's are canonical variables. Their expressions in terms of the original basis are written as

$$U^\alpha = b_i^\alpha u^i.$$

From these we readily get

$$\frac{\partial F}{\partial U^1} \cdots \frac{\partial F}{\partial U^r} = U^1 \cdots U^r = \frac{1}{r!} b^{1 \cdots r}_{i_1 \cdots i_r} u^{i_1} \cdots u^{i_r}, \tag{18.3}$$

and by substitution from (18.3) in which the i's are replaced by j's

$$\frac{\partial F}{\partial u^{i_1}} \cdots \frac{\partial F}{\partial u^{i_r}} = \frac{1}{r!} b^{1 \cdots r}_{i_1 \cdots i_r} b^{1 \cdots r}_{j_1 \cdots j_r} u^{j_1} \cdots u^{j_r}. \tag{18.4}$$

If the form be written

$$F = a_{ij} u^i u^j \qquad (a_{ij} + a_{ji} = 0), \tag{18.5}$$

for any positive integer p we have

$$\frac{\partial F}{\partial u^{i_1}} \cdots \frac{\partial F}{\partial u^{i_p}} = 2^p a_{i_1 j_1} \cdots a_{i_p j_p} u^{j_1} \cdots u^{j_p} = \frac{2^p}{p!} a^{i_1 \cdots i_p}_{j_1 \cdots j_p} u^{j_1} \cdots u^{j_p}.$$

In particular, this formula gives for $p = r$ by comparison with (18.4)

$$2^r a^{i_1\cdots i_r}_{j_1\cdots j_r} = b^{1\cdots r}_{i_1\cdots i_r} b^{1\cdots r}_{j_1\cdots j_r}. \tag{18.6}$$

From (18.2) we have

$$\frac{F^k}{k!} = U^1 \cdots U^r = \frac{1}{r!} b^{1\cdots r}_{i_1\cdots i_r} u^{i_1} \cdots u^{i_r}, \tag{18.7}$$

and from (18.5) for any positive integer q

$$\frac{F^q}{q!} = \frac{1}{q!} a_{i_1 i_2} \cdots a_{i_{2q-1} i_{2q}} u^{i_1} \cdots u^{i_{2q}}.$$

Since the coefficient is invariant under signed permutations of the two types (12), (13)(24), the total number of permutations to which it is necessary to subject the coefficient of this expression in rendering it skew-symmetric by the method of §8 is $(2q)!/(2^q q!)$, and the result of the operation is

$$\frac{F^q}{q!} = \frac{2^q}{(2q)!} c_{i_1\cdots i_{2q}} u^{i_1} \cdots u^{i_{2q}}, \tag{18.8}$$

where

$$c_{i_1\cdots i_{2q}} = a_{i_1 i_2} a_{i_3 i_4} \cdots a_{i_{2q-1} i_{2q}} + \cdots, \tag{18.9}$$

the unwritten terms arising from the first by subjecting its indices to a set of signed permutations which we proceed to describe. A permutation belongs to the set if and only if after its application the pair of integers[12] on every a and the set of q integers occupying the first positions on the a's are in natural relative order.

Putting $q = k$ in (18.8) and comparing with (18.7) we get a means of computing the determinants b directly from the a's:

$$2^k c_{i_1\cdots i_r} = b^{1\cdots r}_{i_1\cdots i_r}. \tag{18.10}$$

Finally, use of this in (18.6) gives

$$a^{i_1\cdots i_r}_{j_1\cdots j_r} = c_{i_1\cdots i_r} c_{j_1\cdots j_r}. \tag{18.11}$$

19. **Applications to matrices and determinants.** The matrix $\| a_{ij} \|$, which incidentally is square, is called *skew-symmetric* for an obvious reason. It is the matrix of the associated set of F. Hence its rank is that of F; namely, $2k$, an *even number*.

The determinant $a^{i_1\cdots i_p}_{i_1\cdots i_p}$ in which the indices are *not summed*, though repeated, is called *principal*. If we put the subscripts equal to the superscripts on the left of (18.6) without summing, we get two important results. First, every principal determinant of order r is a perfect square; second, from (18.3) *there is a non-vanishing principal determinant whose order equals the rank.*

[12] By integers are meant the subscripts appearing on the subscripts.

An arbitrary skew-symmetric determinant of even order can be made a principal determinant of the matrix of a quadratic form. Hence, *every skew-symmetric determinant of even order with elements in* $\mathfrak{R}$ *is a perfect square in* $\mathfrak{R}$.

THEOREM 19.1. *The result of bordering the determinant* $| a^i_j | = a$ *by the column* x^i, z *and the row* y_j, z *is*

$$az - \sum_{i,j} (-1)^i \mathfrak{A}^j_i x^i y_j. \tag{19.1}$$

To prove the theorem, we adjoin a new mark u^0 so that $\mathfrak{R}[u^0, u^1, \cdots, u^n]$ is a Grassmann ring of dimension $n + 1$. Φ being defined by (15.2), we have

$$u^j \frac{\partial \Phi}{\partial u^i} = \delta^j_i \Phi.$$

Ψ being the symbol into which Φ goes under (11.10), we find

$$u^j \frac{\partial \Psi}{\partial v^i} = \mathfrak{A}^j_i .$$

The product

$$(zu^0 + y_j u^j)(v^1 + x^1 u^0) \cdots (v^n + x^n u^0),$$

whose coefficient by (11.17) is precisely the determinant sought, can be rewritten

$$u^0 \left[z\Psi + (-1)^{i-1} y_j u^j x^i \frac{\partial \Psi}{\partial v_i} \right].$$

The coefficient of this last expression is precisely (19.1).

CHAPTER IV

DIFFERENTIAL RINGS

The present chapter introduces three differential and three integral assumptions. From them results a body of theorems concerning forms in relation to their differentials. At the basis of all the properties discussed is the notion of a passive pfaffian system.

20. **The differential assumptions.** We assume

D_1. *There exists an integrity domain $\mathfrak{R}'$, containing $\mathfrak{R}$, such that to every symbol a of $\mathfrak{R}$ there corresponds in $\mathfrak{R}'[u]$ a unique form*

$$a' = (\delta_i a)u^i, \tag{20.1}$$

which is zero or linear, called the differential of a. The differentials of sums and products in $\mathfrak{R}[u]$ satisfy

$$(a + b)' = a' + b', \tag{20.2}$$

$$(ab)' = a'b + b'a. \tag{20.3}$$

This rather complex relation of $\mathfrak{R}'$ to $\mathfrak{R}$ will be described concisely by saying that "$\mathfrak{R}$ is differentiable in $\mathfrak{R}'$."

The substitutions $a = 1$, $b = 0$ in (20.2) and $a = 1$, $b = 1$ in (20.3) give

$$0' = 0, \qquad 1' = 0. \tag{20.4}$$

The symbol $\delta_i a$, which belongs to $\mathfrak{R}'$, is called *the differential coefficient of a of index i*. This delta is not to be confused with the Kronecker delta (§5), which has two indices. Equating coefficients in (20.2) and (20.3) gives

$$\delta_i(a + b) = \delta_i a + \delta_i b, \qquad \delta_i(ab) = a\delta_i b + b\delta_i a. \tag{20.5}$$

The differential of the form (8.1) is defined as

$$F' = a'_{i_1 \cdots i_p} u^{i_1} \cdots u^{i_p},$$

or in words, the differential of a form of degree $p > 0$ is found by replacing each coefficient by its differential. This statement assumes, of course, that the coefficient is written in the leading position. Relation (20.2) is seen to hold if a and b are forms of degree p. In order to apply to forms of degree p and q respectively, relation (20.3), however, must be modified to

$$(ab)' = a'b + (-1)^{pq} b'a, \tag{20.6}$$

the sign of the second term being minus only if both a and b have odd degree. It is clear that the following theorem is true.

THEOREM 20.1. *If F belongs to $\mathfrak{R}[u]$, F' belongs to $\mathfrak{R}'[u]$.*

Because of this, $\mathfrak{R}[u]$ is called *differentiable in* $\mathfrak{R}'[u]$.

Writing u^i as the linear form $1u^i$, we see that *the marks have zero differentials*.

A symbol of $\mathfrak{R}$ whose differential is zero is called a *constant*. Use of (20.2), (20.3), (20.4) proves

THEOREM 20.2. *The set of all constants is a subring of* $\mathfrak{R}$ *and includes the symbols* $0, \pm 1, \pm 2, \cdots$.

The foregoing result can be generalized. Let $i_1 \cdots i_k i_{k+1} \cdots i_n$ represent the symbols $1, 2, \cdots, n$ written in some order. The symbols a which are in $\mathfrak{R}$ and which satisfy

$$\delta_{i_{k+1}} a = 0, \quad \cdots, \quad \delta_{i_n} a = 0$$

can be proved to form a ring, which will be denoted by $\mathfrak{R}_{i_{k+1} \cdots i_n}$. It is important to note that this ring depends upon the basis to which $\mathfrak{R}[u]$ is referred. The indices on $\mathfrak{R}$ will always be interpreted as referring to the basis appearing in the brackets. The ring $\mathfrak{R}'$ can be employed to define the differentials for the ring of dimension k, $\mathfrak{R}_{i_{k+1} \cdots i_n}[u^{i_1}, \cdots, u^{i_k}]$, by taking for the differential coefficients the symbols $\delta_{i_1} a, \cdots, \delta_{i_k} a$ from $\mathfrak{R}'$. When this is done, we have

THEOREM 20.3. *The differential of a form, when it is regarded as belonging to* $\mathfrak{R}_{i_{k+1} \cdots i_n}[u^{i_1}, \cdots, u^{i_k}]$, *is the same as its differential as a form of* $\mathfrak{R}[u]$.

The ring involved will be called a *differential subring* of $\mathfrak{R}[u]$ because of the property stated in the above theorem.

THEOREM 20.4. *If* F *and* F' *belong to* $\mathfrak{R}[u^1, \cdots, u^{n-1}]$, *they belong to* $\mathfrak{R}_n[u^1, \cdots, u^{n-1}]$.

To prove the last stated result, write the standard form of F as

$$F = a_\alpha m^\alpha,$$

where the a's belong to $\mathfrak{R}$ and the m's are dissimilar unit monomials. By hypothesis, no m contains u^n. Consequently, direct calculation gives

$$\partial F' / \partial u^n = (\delta_n a_\alpha) m^\alpha.$$

Since the terms in the right member are dissimilar, Theorem 7.1 shows that $\delta_n a_\alpha = 0$, and the result is proved. It remains valid, of course, if u^n is replaced by any other mark.

The second differential assumption is

D_2. *The ring* $\mathfrak{R}'$ *is differentiable in a commutative ring* $\mathfrak{R}''$, *and, if* a *belongs to* $\mathfrak{R}[u]$,

$$(a')' = 0.[13] \tag{20.7}$$

[13] H. W. Raudenbush, Jr. has used definitions like D_1, D_2, for partial differentiation in a ring $\mathfrak{R} = \mathfrak{R}'$ of functions. Cf. Bulletin of the American Mathematical Society, vol. 40 (1934), p. 715.

Interpreting the a in (20.7) as belonging to $\mathfrak{R}$, and equating to zero the skew-symmetric coefficient, we get

$$\delta_i(\delta_j a) - \delta_j(\delta_i a) = 0,$$

a relation equivalent to (20.7). We shall write it

$$\delta_i \delta_j - \delta_j \delta_i = 0. \tag{20.8}$$

Since $\mathfrak{R}'$ is an integrity domain, it can be imbedded [23, I, 47] in a commutative field $(\mathfrak{R}')^*$. It is readily proved that $\mathfrak{R}^*$ is differentiable in $(\mathfrak{R}')^*$. Hence we may put $(\mathfrak{R}^*)' = (\mathfrak{R}')^*$.

21. The first and second integral assumptions. We assume

I_1. *To every symbol a of $\mathfrak{R}$ and every symbol i from $1, 2, \cdots, n$ there corresponds in $\mathfrak{R}$ at least one symbol $\sigma^i a$ whose differential coefficient of index i is a.*

Clearly we have from the definition

$$\delta_i \sigma^i a = a \qquad (i \text{ fixed}). \tag{21.1}$$

THEOREM 21.1. *There exists in $\mathfrak{R}'[u]$ a form φ whose differential is a prescribed form F which is of degree greater than zero and belongs to $\mathfrak{R}[u]$, if and only if $F' = 0$.*

Only the sufficiency of the condition remains to be proved. The proof will be accomplished by induction on the number of the marks.

If there is only one mark u^1, we have $F = au^1$, where a belongs to $\mathfrak{R}$. If we put $\varphi = \sigma^1 a$, then by definition

$$\varphi' = \delta_1(\sigma^1 a)u^1 = au^1 = F,$$

and the theorem is proved for $n = 1$.

As in §13, any form in a ring with n marks can be written

$$F = f + u^n g,$$

where f and g belong to $\mathfrak{R}[u^1, \cdots, u^{n-1}]$. Let

$$g = a_{\alpha_1 \cdots \alpha_p} u^{\alpha_1} \cdots u^{\alpha_p} \qquad (\alpha_1, \cdots, \alpha_p = 1, \cdots, n-1)$$

and

$$h = (\sigma^n a_{\alpha_1 \cdots \alpha_p}) u^{\alpha_1} \cdots u^{\alpha_p}.$$

Then

$$h' = (\delta_{\alpha_0} \sigma^n a_{\alpha_1 \cdots \alpha_p}) u^{\alpha_0} \cdots u^{\alpha_p} + u^n g,$$

so that $F - h'$ belongs to $\mathfrak{R}'[u^1, \cdots, u^{n-1}]$. Since $(F - h')' = 0$, Theorem 20.4 shows that $F - h'$ belongs to $\mathfrak{R}_n'[u^1, \cdots, u^{n-1}]$. Because of Theorem 20.3 its differential as a member of that ring is also zero. Assuming the theorem for rings with $n - 1$ marks, we therefore conclude the existence of a form k in

$\mathfrak{R}_n'[u^1, \cdots, u^{n-1}]$ satisfying $k' = F - h'$. With k so determined, $\varphi = h + k$ is clearly a solution of

$$\varphi' = F, \tag{21.2}$$

so that the induction is complete.

If φ_1 and φ_2 both satisfy (21.2), then $(\varphi_1 - \varphi_2)' = 0$. Consequently we have

THEOREM 21.2. *A general solution of* (21.2) *is* $\varphi_1 + \varphi_2$, *where* φ_1 *is any particular solution and* φ_2 *is any form of degree one less than* F *satisfying* $\varphi_2' = 0$.

Because of Theorem 21.1 each of the equations

$$x'^i = u^i \tag{21.3}$$

has a solution x^i in $\mathfrak{R}'$, that is, the marks are differentials, and because of Theorem 21.2 a general solution of (21.3) is $x^i + c^i$, where x^i represents a particular solution and each c^i is an arbitrarily chosen member of the ring of constants. We fix upon a particular solution of (21.3) and call the x's so obtained the *independent variables*. Because of (6.3) they satisfy

$$x'^1 \cdots x'^n \neq 0. \tag{21.4}$$

The x's, which so far are only known to be in $\mathfrak{R}'$, can be proved to belong to $\mathfrak{R}$ by applying the method of proof used for Theorem 21.1 directly to equations (21.3) taken separately.

The ring $\mathfrak{R}_{i_{k+1}\cdots i_n}$ can now be denoted by $\mathfrak{R}_{x^{i_{k+1}}\cdots x^{i_n}}$ when indicating the marks of differentiation seems desirable.

A symbol which is a function of $x^1, \cdots, x^n$ but of no subset of those variables is called a *proper function* of $x^1, \cdots, x^n$.

The second integral assumption is

I_2. *Every symbol of* $\mathfrak{R}$ *is a function of* $x^1, \cdots, x^n$. *A symbol of* $\mathfrak{R}$ *is a proper function of* $x^{i_1}, \cdots, x^{i_n}$ *if and only if it belongs to* $\mathfrak{R}_{i_{r+1}\cdots i_n}$, *where* $i_1 \cdots i_n$ *is a permutation of* $1 \cdots n$, *but to none of its subrings.*

22. **Differential coefficients of higher order.** The differential coefficient defined in §20 is said to be of the first order. We shall in the future call it by the usual name, partial derivative, and put $\delta_i = \partial/\partial x^i$. It may be possible to define derivatives of higher order which satisfy the recursion formulas

$$\mathfrak{R}^k = (\mathfrak{R}^{k-1})', \qquad \delta_{i_1 i_2 \cdots i_k} = \delta_{i_1}(\delta_{i_2 \cdots i_k}) \qquad (k = 1, 2, \cdots, N). \tag{22.1}$$

To insure this, we make at times a third differential assumption

D_3. *The ring* $\mathfrak{R}$ *is differentiable* N *times in the ring* $\mathfrak{R}^N$, *where* N *is the order of the highest derivative appearing in the discussion.*

Because of (20.8) the symbol δ in (22.1) is symmetric in its subscripts. Hence to identify it we need only give the number j_1 of its indices which are equal to 1, the number j_2 which are equal to 2, etc. With δ we accordingly associate the

symbol $j_1 \cdots j_n$ which is sufficient to identify it. This establishes between commutative monomials (§36) and derivatives a one-to-one correspondence, which will be of importance later. For the present, we remark that it is often convenient to make the abbreviation $m = x_1^{i_1} \cdots x_n^{i_n}$ and put

$$\frac{\partial}{\partial m} = \frac{\partial^{i_1+\cdots+i_n}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}. \tag{22.2}$$

An important property of these symbols $i_1 \cdots i_n$, and consequently of the derivatives they represent, is contained in

THEOREM 22.1. *A sequence of symbols $j_1 \cdots j_n$, in which each j is a non-negative integer, and for which at least one of the differences*

$$i_1 - j_1, \ \cdots, \ i_n - j_n \tag{22.3}$$

is negative if $j_1 \cdots j_n$ precedes $i_1 \cdots i_n$, is necessarily finite.

The proof is by induction on n. Fix $j_1 \cdots j_n$ in (22.3) as the first symbol of the sequence. The result is true for $n = 1$ because there are only j_1 integers i_1 satisfying $0 \leqq i_1 < j_1$.

Let the symbols of the sequence be put into classes M_{kl} ($l = 0, 1, \cdots, j_k$; $k = 1, 2, \cdots, n$), which in general overlap, as follows. The symbol $i_1 \cdots i_n$ is placed in M_{kl} if $i_k = l$. Since $i_k < j_k$ for at least one k, every symbol of the sequence is placed in at least one M. The symbols in M_{kl} all have the kth index equal. If abstraction is made of that index, and they are conceived as being in the same relative order as the symbols in the original sequence, they are seen to form a sequence satisfying the hypothesis of the theorem. If the theorem is assumed for $n - 1$ indices, it follows that each M_{kl} contains a finite number of symbols. Since there is a finite number of sets M_{kl}, the number of symbols in the sequence in n indices is finite, and the induction is complete.

The foregoing theorem amounts to: *a sequence of monomials no one of which is a multiple of any of its predecessors is necessarily finite.*

23. Indirect differentiation. If F is a function of x^i, y^α belonging to the ring $\mathfrak{R}[x'^1, \cdots, x'^n; y'^1, \cdots, y'^r]$, its differential is given by

$$F' = F_{x^i} x'^i + F_{y^\alpha} y'^\alpha, \tag{23.1}$$

where subscripts are used to denote derivatives in the above ring. If now the y's are identified with certain functions of the x's belonging to $\mathfrak{R}$, F becomes a function of the x's, whose derivatives

$$\delta_i F = F_{x^i} + F_{y^\alpha} \delta_i y^\alpha, \tag{23.2}$$

where the δ's refer to $\mathfrak{R}'[x']$.

24. Transformation of the marks. It is now convenient to write a transformation of marks as

$$u^i = A_j^i \bar{u}^j, \tag{24.1}$$

where the A's belong to $\mathfrak{R}$ and where, of course, the bar does not have the significance it has in Chapter II. This transformation applied to the u's in (20.1) gives the linear (or zero) form

$$(\delta_i a)A^i_j \bar{u}^j. \tag{24.2}$$

If this is called $\bar{a}'$, clearly we have

$$\bar{a}' + \bar{b}' = (\delta_i a + \delta_i b)A^i_j \bar{u}^j = (\overline{a + b})',$$

$$\bar{a}'b + \bar{b}'a = (b\delta_i a + a\delta_i b)A^i_j \bar{u}^j = (\overline{ab})',$$

that is, equations analogous to (20.2) and (20.3) are satisfied. As far as those conditions are concerned, therefore, (24.2) serves to define the differential of a in the ring $\mathfrak{R}'[\bar{u}]$. Let this definition be adopted.

In accordance with (24.2) the differential coefficients $\bar{\delta}_i a$ with respect to the new marks satisfy

$$\bar{\delta}_j a = (\delta_i a)A^i_j, \tag{24.3}$$

relations which will be written

$$\bar{\delta}_j = A^i_j \delta_i .$$

By (12.5) the transformation of the δ's is contragredient to that of the marks. Moreover, if a belongs to $\mathfrak{R}'$, the quantities defined by (24.3) belong to $\mathfrak{R}''$ so that $\mathfrak{R}'$ is differentiable in $\mathfrak{R}''[\bar{u}]$ under definition (24.2), and the first part of D_2 is satisfied by the definition of differentiation adopted for $\mathfrak{R}[u]$.

Since the marks u must have zero differentials, however, a transformation (24.1) must satisfy other conditions. Expressing that (24.1) has zero differential in $\mathfrak{R}'[u]$ gives

$$\delta_k A^i_j A^k_l - \delta_j A^i_k A^k_j = 0. \tag{24.4}$$

Applying formula (24.2) for the computation of its own differential, we find

$$\delta_k[(\delta_i a)A^i_j]A^k_l \bar{u}^l \bar{u}^j.$$

Use of (20.5), (20.8) reduces this to

$$(\delta_i a)(\delta_k A^i_j)A^k_l \bar{u}^l \bar{u}^j. \tag{24.5}$$

Substitution from (24.4) in (24.5) gives zero. Hence (20.7) is satisfied by all linear forms in $\mathfrak{R}'[\bar{u}]$ and therefore by all forms in $\mathfrak{R}'[\bar{u}]$. Hence D_2 is completely satisfied by $\mathfrak{R}[\bar{u}]$.

The correspondence established by (24.1) and (24.3) between the particular solutions of (21.3) and the analogous equations for $\mathfrak{R}[\bar{u}]$ is called a *transformation of independent variables*:

$$x^i \rightarrow \bar{x}^i. \tag{24.6}$$

The $\bar{x}$'s may, of course, be assigned as any symbols of $\mathfrak{R}$ satisfying the condition like (21.4), namely,

$$\bar{x}'^1 \cdots \bar{x}'^n \neq 0, \tag{24.7}$$

and the transformation of marks (24.1) is then completely determined.

The transformation obtained by taking the $\bar{x}$'s in (24.6) as arbitrary symbols, whose differentials do not necessarily satisfy (24.7), is also of importance. The corresponding transformation (24.1) is

$$\bar{x}'^i = (\bar{\delta}_j x^i)x'^j. \tag{24.8}$$

Let the combined substitution (24.6), (24.8) convert a form F into F^* and its differential F' into $(F')^*$. We then have

$$(F^*)' = (F')^*, \tag{24.9}$$

a useful formula, which is proved as follows. From (8.1)

$$F^* = \overset{*}{a}_{i_1 \cdots i_p}[A^{i_1}_{\alpha_1} \cdots A^{i_p}_{\alpha_p}\bar{u}^{\alpha_1} \cdots \bar{u}^{\alpha_p}], \qquad F' = da_{i_1 \cdots i_p}u^{i_1} \cdots u^{i_p}.$$

Now since

$$u^{i_1} \cdots u^{i_p} = A^{i_1}_{\alpha_1} \cdots A^{i_p}_{\alpha_p}\bar{u}^{\alpha_1} \cdots \bar{u}^{\alpha_p},$$

we have that the part of F^* written in brackets can be treated as a constant in the differentiation. Hence

$$(F^*)' = d\overset{*}{a}_{i_1 \cdots i_p}A^{i_1}_{\alpha_1} \cdots A^{i_p}_{\alpha_p}\bar{u}^{\alpha_1} \cdots \bar{u}^{\alpha_p}.$$

Also

$$(F')^* = (da_{i_1 \cdots i_p})^* A^{i_1}_{\alpha_1} \cdots A^{i_p}_{\alpha_p}\bar{u}^{\alpha_1} \cdots \bar{u}^{\alpha_p}.$$

The desired formula is seen to be true provided

$$d\overset{*}{a}_{i_1 \cdots i_p} = (da_{i_1 \cdots i_p})^*. \tag{24.10}$$

This is the particular case of (24.9), where F is of degree zero, and is true because it amounts to the definition (24.2).

An application of (24.9) is the following. By the substitution

$$T\colon x^{i_1} = x^{i_1}, \quad \cdots, \quad x^{i_r} = x^{i_r}, \quad x^{i_{r+1}} = c^{i_{r+1}}, \quad \cdots, \quad x^{i_n} = c^{i_n}, \tag{24.11}$$

where the c's are constants from the scope of the corresponding variables, the function $f(x^1, \cdots, x^n)$ is converted into a function belonging to $\mathfrak{R}_{i_{r+1} \cdots i_n}$. When (24.9) is applied to this case and coefficients of the dx's are equated, there results

$$\frac{\partial}{\partial x^i}(Tf) = T\left(\frac{\partial f}{\partial x^i}\right) \qquad (i = i_1, \cdots, i_r). \tag{24.12}$$

25. The third integral assumption. It has been seen in §11 that a set of linear forms of rank r determines a subring $\mathfrak{R}[v^1, \cdots, v^r]$. When $\mathfrak{R}[u]$ satisfies the assumptions made so far and $\mathfrak{R} = \mathfrak{R}^*$, the totality of linear forms in the

subring is called a *pfaffian system of rank* r. Any set of linear forms of rank r in the system is called a *basis* of the pfaffian system.

In accordance with Theorem 15.3, any monomial form determines a pfaffian system comprising all its linear factors. If $\omega^1, \cdots, \omega^r$ is a basis of the pfaffian system, the form can be written

$$F = a\omega^1 \cdots \omega^r.$$

A pfaffian system is *passive* if it has a basis belonging to $\Re[u]$ and composed of differentials, called a *differential basis.* Since a system of rank n has the marks for basis, we have

THEOREM 25.1. *Every pfaffian system of rank* n *in* $\Re[u^1, \cdots, u^n]$ *is passive.*

If $v^1, \cdots, v^k$ are independent differentials of a pfaffian system S, by Theorem 11.3 they can be made the first k members of a basis for S. Furthermore, if S is passive and has for basis the differentials $u^1, \cdots, u^r$, there must exist $r - k$ u's whose product by the k v's is not zero; otherwise the rank of the u's by Theorem 11.5 would be less than r. Hence

THEOREM 25.2. *Any* k *independent differentials of a pfaffian system can be made part of a basis for the system. If the system is passive, they can be made part of a differential basis.*

The third integral assumption is

I_3. *Every pfaffian system of rank* $n - 1$ *in a ring of dimension* n *is passive.*

A Grassmann ring satisfying the five assumptions of the present chapter is called a *differential ring.* The existence of such rings will be proved later by means of consistency examples. For the moment our attention will be confined to developing the consequences of our assumptions, the first result being

THEOREM 25.3. *There exists a transformation of variables such that a given pfaffian system of rank less than* n *belongs to* $\Re[\bar{u}^1, \cdots, \bar{u}^{n-1}]$; *in other words, the system can be expressed in fewer than* n *differentials.*

If the rank of the system is r, augment it by $n - r - 1$ linear forms chosen, as in §11, so that the rank of the whole is $n - 1$. The augmented system has by I_3 a basis consisting of $n - 1$ differentials $\bar{u}^1, \cdots, \bar{u}^{n-1}$, in terms of which every form of the system, in particular those of the original pfaffian system, can be expressed. The product of these $\bar{u}$'s by at least one u is not zero; otherwise the rank of the n u's would be $n - 1$ by Theorem 11.5. Such a u can be used with the $n - 1$ $\bar{u}$'s to define a transformation of independent variables (§24), and the theorem is proved.

26. The characteristic system. The associated set (§14) of the forms F^α and F'^α determines a pfaffian system, called the *characteristic system* of the set of forms F^α. Its rank c is called the *class* of the set.

If $c < n$, the characteristic system belongs to an $\Re[\bar{u}^1, \cdots, \bar{u}^{n-1}]$ by Theorem

25.3. Hence (Theorem 14.2) the forms F^α and F'^α belong to $\mathfrak{R}[\bar{u}^1, \cdots, \bar{u}^{n-1}]$. By Theorems 20.4 and 20.3 F^α and F'^α belong to $\mathfrak{R}_n[\bar{u}^1, \cdots, \bar{u}^{n-1}]$ and F'^α is the differential of F^α in that ring. By Theorem 11.4 the class is the same whether the forms be considered as belonging to $\mathfrak{R}[u]$ or to $\mathfrak{R}_n[\bar{u}]$. Hence if $c < n - 1$, the process above can be repeated. It terminates when the dimension of the ring is the class c. The forms F^α accordingly belong to an $\mathfrak{R}_{c+1\cdots n}[v^1, \cdots, v^c]$. Since the characteristic system is thus a pfaffian system of rank c in a ring of dimension c, it is passive by Theorem 25.1.

Symbols y such that $y'^\alpha = v^\alpha$ are called *characteristic variables* for the set of forms F^β.

Notions parallel to the above will now be developed for a pfaffian system. If S is a pfaffian system with rank r and basis ω^α, the associated set of the forms

$$\omega^\alpha, \quad \Omega\omega'^\alpha, \tag{26.1}$$

where

$$\Omega = \omega^1 \cdots \omega^r, \tag{26.2}$$

defines the *characteristic system* C of S, and again the rank c of C is called the *class* of S. It is clear that C contains S and that $c \geqq r$.

If the basis of S is transformed by

$$\bar{\omega}^\alpha = b^\alpha_\beta \omega^\beta \qquad (\alpha, \beta = 1, 2, \cdots, r), \tag{26.3}$$

that is, by a transformation (11.10), use of (20.6) gives the formulas

$$\bar{\omega}'^\alpha = b'^\alpha_\beta \omega^\beta + b^\alpha_\beta \omega'^\beta. \tag{26.4}$$

Moreover,

$$\bar{\Omega} = b\Omega, \tag{26.5}$$

where b is the determinant of order r formed from b^α_β. Multiplication of (26.4) and (26.5) gives

$$\bar{\Omega}\bar{\omega}'^\alpha = bb^\alpha_\beta \Omega\omega'^\beta. \tag{26.6}$$

These equations can be solved for the unbarred symbols on the right by multiplication with B^γ_α and use of relations analogous to (11.13). Hence (26.1) undergo a linear transformation of forms (§14), and Theorem 14.5 gives

THEOREM 26.1. *The characteristic system and class of a pfaffian system are invariant under transformation of its basis.*

If the class $c < n$, as in the case of a set of forms, we infer the existence of an $\mathfrak{R}[\bar{u}^1, \cdots, \bar{u}^{n-1}]$ to which (26.1) belong. Let the basis of S be normalized (§11) so that it has the appearance

$$\omega^\alpha = \bar{u}^\alpha + b^\alpha_\lambda \bar{u}^\lambda \qquad (\alpha = 1, 2, \cdots, r; \lambda = r + 1, \cdots, n - 1).$$

Since $\bar{u}^n$ has zero coefficients in (26.1), the same is true of the products

$$\Omega \frac{\partial \Pi}{\partial \bar{u}^\lambda} \omega'^\alpha = (-1)^{n-r+1} \delta_n b_\lambda^\alpha \bar{u}^1 \cdots \bar{u}^n,$$

where

$$\Pi = \bar{u}^{r+1} \cdots \bar{u}^{n-1}.$$

Hence $\delta_n b_\lambda^\alpha = 0$ and the normalized basis belongs to $\mathfrak{R}_n[\bar{u}^1, \cdots, \bar{u}^{n-1}]$. If $c < n - 1$, the argument can be repeated, so that we infer the existence of a ring $\mathfrak{R}_{c+1,\cdots,n}[v^1, \cdots, v^c]$ containing a basis of the characteristic system.

THEOREM 26.2. *The characteristic system of a set of forms or of a pfaffian system is passive. Its rank, the class, is the dimension of the minimum differential subring containing the set of forms or a basis of the pfaffian system.*

Variables whose differentials are the v's encountered above are again called *characteristic.*

If the ω's in (26.1) are a differential basis of a passive S, the conditions

$$\Omega\omega'^\alpha = 0 \tag{26.7}$$

are satisfied. Conversely, if these products vanish, the class (= rank of C) is r. Since C contains S, the two coincide, and S is passive because C is by Theorem 26.2. Hence we have a result of fundamental importance in the study of pfaffian systems:

THEOREM 26.3. *A pfaffian system is passive if and only if conditions* (26.7) *are satisfied.*

The condition just given can be paraphrased "the rank equals the class," or "the system coincides with its characteristic system," or "the differential of every form of the ideal determined in $\mathfrak{R}[u]$ by the system belongs to the ideal."

From the definition (26.2) obviously results

$$\Omega\omega^\alpha = 0. \tag{26.8}$$

Forming the differential of this gives

$$\Omega'\omega^\alpha = (-1)^{r-1}\,\Omega\omega'^\alpha. \tag{26.9}$$

Hence conditions (26.7) are equivalent to $\Omega'\omega^\alpha = 0$, which by Theorem 15.2 can be written

$$\Omega' = \varphi\Omega. \tag{26.10}$$

This form of the condition for a passive system will be found very useful.

The case $\varphi = 0$ in (26.10) is of particular importance. We shall prove

THEOREM 26.4. *A monomial form can be written*

$$\Omega = u^1 \cdots u^r \tag{26.11}$$

if and only if

$$\Omega' = 0. \tag{26.12}$$

When the condition is satisfied, the pfaffian system is passive, and any $r - 1$ *independent differentials belonging to it can be made the* $u^2, \cdots, u^r$ *in* (26.11).

Only the sufficiency of (26.12) needs to be proved. From the passivity of the corresponding pfaffian system we may write (see (11.10))

$$\Omega = au^1 \cdots u^r. \tag{26.13}$$

Condition (26.11) shows that

$$\delta_\lambda a = 0 \qquad (\lambda = r + 1, \cdots, n).$$

If we put $\bar{x}^1 = \sigma^1 a$, then

$$\bar{u}^1 = au^1 + \delta_\alpha(\sigma^1 a)u^\alpha \qquad (\alpha = 2, \cdots, r).$$

Substitution in (26.13) gives

$$\Omega = \bar{u}^1 u^2 \cdots u^r.$$

The rest of the statement follows from Theorem 25.2.

27. **The canonical form of a pfaffian.** A linear differential form ω is called a *pfaffian.* It is to be distinguished from the pfaffian system of rank one having ω for basis because the latter comprises all pfaffians $a\omega$, where $a \neq 0$ is an arbitrary symbol of $\mathfrak{R}$.

Let us make in the notation of the algebraic Theorem 17.1 the substitution of ω', ω, $g^1, \cdots, g^l$ for F, $u^1, u^2, \cdots, u^h$, respectively, and set

$$\Omega = \omega'^k \omega g^1 \cdots g^l.$$

Ω is monomial by Theorem 17.1. Forming the differential on the assumption that the g's are differentials gives

$$\Omega' = \omega'^{k+1} g^1 \cdots g^l.$$

Substitution in (17.10) shows that (26.10) holds. Hence the u^{k+1} of Theorem 17.1 can be taken as a *differential* g^{l+1}.

Suppose ρ defined by

$$\omega'^\rho \omega \neq 0, \qquad \omega'^{\rho+1}\omega = 0, \tag{27.1}$$

and let ω be reduced to (17.8) by the method of §17, differentials being employed for the successive u's. The process can be carried one stage farther than is done there, the last conditions being that

$$g^1 \cdots g^{h+1} \neq 0, \qquad \omega g^1 \cdots g^{h+1} = 0.$$

The result is

$$\omega = f_\alpha g^\alpha \qquad (\alpha = 1, 2, \cdots, \rho + 1). \tag{27.2}$$

Let $\omega'^{\rho+1} \neq 0$, and $g^\alpha = x'^\alpha$. Reference to (21.4) shows that (27.2) belongs to a differential ring for which independent variables are f_α, x^α. We rewrite

(27.2) as

$$\omega = p_\alpha u^\alpha, \tag{27.3}$$

taking the g's as marks. The independent variables are then p, x in the subring of dimension $2\rho + 2$ to which ω belongs. The characteristic system of the form ω comprises the symbols p'_α, u^α so that the class of ω is $2\rho + 2$. The right member of (27.3) is called the canonical form of ω. The conditions for reducibility to it are

$$\omega'^{\rho+1} \neq 0, \qquad \omega'^{\rho+1}\omega = 0. \tag{27.4}$$

The first condition of (27.1) is omitted, for by Theorem 17.1 the first condition of (27.4) implies that ω'^ρ has no linear divisor and in particular, therefore, is not divisible by ω.

If $\omega'^{\rho+1} = 0$, by Theorem 17.1 ω'^ρ is monomial. Moreover, it satisfies (26.12). Hence it can be written in the form (26.11). The same is true of $\omega'^\rho\omega$, and since that form is divisible by ω'^ρ, Theorem 26.4 shows the existence of a z in $\mathfrak{R}$ satisfying

$$\omega'^\rho(\omega - z') = 0. \tag{27.5}$$

With z so determined, since $\omega' = (\omega - z')'$, the form $\omega - z'$ satisfies conditions (27.4) in which $\rho + 1$ is replaced by ρ. Therefore

$$\omega = z' + p_\alpha u^\alpha \qquad (\alpha = 1, 2, \cdots, \rho). \tag{27.6}$$

The symbols z, p_α, x^α are independent variables because of (27.1), and the class of ω is found to be $2\rho + 1$. The conditions for reducibility to (27.6) are

$$\omega'^\rho\omega \neq 0, \qquad \omega'^{\rho+1} = 0. \tag{27.7}$$

In either case, because of inequation (27.1), at least one of the f's in (27.2) is not zero. If we suppose $f_{\rho+1} \neq 0$, ω can be displayed with a slight change of notation as

$$\omega = f_{\rho+1}(z' + p_\alpha u^\alpha) \qquad (\alpha = 1, 2, \cdots, \rho).$$

Hence the pfaffian system with ω for basis also has the basis

$$z' + p_\alpha u^\alpha \qquad (\alpha = 1, 2, \cdots, \rho), \tag{27.8}$$

which will be called a *canonical basis*. The class of the pfaffian system is readily found to be $2\rho + 1$. The conditions that ω possess the basis (27.8) are, of course, (27.1).

It is convenient to recapitulate the above results by means of the following table, in which α has the range $1, 2, \cdots, \rho$:

		Class	Canonical form	Conditions
Pfaffian form	ω	2ρ	$p_\alpha u^\alpha$	$\omega'^\rho \neq 0,\ \omega'^\rho\omega = 0$
Pfaffian form	ω	$2\rho + 1$	$z' + p_\alpha u^\alpha$	$\omega'^\rho\omega \neq 0,\ \omega'^{\rho+1} = 0$
Pfaffian system	ω	$2\rho + 1$	$z' + p_\alpha u^\alpha$	$\omega'^\rho\omega \neq 0,\ \omega'^{\rho+1}\omega = 0.$

In all three cases, reduction to canonical form involves determination of the symbols in a form like (27.2). We write

$$\omega_m = \omega'^m \omega,$$

and display below the conditions which the g's must be determined successively to satisfy. A g without superscript denotes the symbol to be determined.

$$\begin{array}{rlcc}
\multicolumn{2}{c}{\text{System}} & & \text{Solution} \\
\omega_\rho g &= 0, & g \neq 0 & \overset{1}{g} \\
\omega_{\rho-1}\overset{1}{g}g &= 0, & \overset{1}{g}g \neq 0 & \overset{2}{g} \\
\vdots & & \vdots & \vdots \\
\omega\overset{1}{g}\cdots\overset{\rho}{g}g &= 0, & \overset{1}{g}\cdots\overset{\rho}{g}g \neq 0 & \overset{\rho+1}{g}.
\end{array}$$

Each g is thus required to be a differential of a certain passive pfaffian system, which is obtained by writing the associated set of the corresponding form

$$\omega_\rho, \quad \omega_{\rho-1}\overset{1}{g}, \quad \omega_{\rho-2}\overset{1}{g}\overset{2}{g}, \quad \cdots, \quad \omega\overset{1}{g}\cdots\overset{\rho}{g}.$$

Multiplication of (27.2) on the left by $\overset{1}{g}\cdots\overset{\alpha-1}{g}$ and on the right by $\overset{\alpha+1}{g}\cdots\overset{\rho+1}{g}$ gives

$$\omega\frac{\partial G}{\partial \overset{\alpha}{g}} = f_\alpha G, \tag{27.9}$$

where

$$G = \overset{1}{g}\cdots\overset{\rho+1}{g}. \tag{27.10}$$

When the g's have once been determined, the f's are determined by equating coefficients in (27.9). It is, of course, only necessary to compare the coefficients of a single monomial on both sides of (27.9) in order to get f_α.

CHAPTER V

COMMUTATIVE MONOMIALS AND POLYNOMIALS

In this chapter are stated in a form adapted to the needs of later chapters certain properties of commutative polynomial rings.

28. **Factorization.** If three symbols of $\mathfrak{R}$ are connected by the equation

$$f = gh, \tag{28.1}$$

f is said to *have as a factor* each symbol on the right.

If the f in (28.1) is the identity symbol 1 of multiplication, g is called the *inverse* of h, and vice versa. A symbol having an inverse is a *unit*.[14] The set of all units is a group under multiplication and belongs to $\mathfrak{R}$.

If h in (28.1) is a *unit*, g is an *improper* factor of f, and the factorization is *trivial*. In this case, the relation between f and g is reciprocal and they are called *associates*.[15]

If neither symbol on the right of (28.1) is a unit, both g and h are *proper factors* of f. The symbol f always has the improper factors f and $-f$. If it has only improper factors, it is *prime*. If it has a proper factor, it is *composite*.

If g or h in (28.1) is composite, we replace it by a non-trivial factorization. Let this process be repeated. We assume the following *unique factorization theorem*.

A_7. *Every symbol f of $\mathfrak{R}$ can be expressed as the product of a finite number of prime factors*

$$f = f_1 f_2 \cdots f_e. \tag{28.2}$$

If a symbol is expressed in two ways as the product of prime factors, the non-unit factors can be put in one-to-one correspondence so that corresponding factors are associates.

By collecting together in the right member of (28.2) the f_i's which are associates, we may write

$$f = A f_1^{d_1} f_2^{d_2} \cdots f_s^{d_s}, \tag{28.3}$$

where A is a unit, no two f_i's are associates and each d_i is a positive integer called the *multiplicity* of the corresponding f_i. From A_7, it follows that any factor h of f can be written

$$h = B f_1^{e_1} f_2^{e_2} \cdots f_s^{e_s}, \tag{28.4}$$

[14] This term is not to be confused with unit monomial or unit polynomial.

[15] No confusion need arise from our employing this term in a sense different from that which it has in §12, where the pairs of correspondents in two rings are called associates.

where B is a unit and each e_i is a non-negative integer not exceeding the corresponding d_i.

By use of A_7 we readily establish the following

THEOREM 28.1. *If a product fg has a prime factor h, either f contains h or g does.*

From (28.3) and (28.4) we readily find, if f and g are any two symbols of $\mathfrak{R}$, that

$$f = hf_1, \qquad g = hg_1, \tag{28.5}$$

where f_1 and g_1 have no non-unit prime factor in common. h is called a *common* of f and g. h is determined except for a unit factor and has as a factor every factor common to f and g. If h is a unit, f and g are called *relatively prime.*

These definitions admit immediate extension to any finite set of symbols $f_1, \cdots, f_e$. Thus

$$f_1 = hg_1, \qquad f_2 = hg_2, \qquad \cdots, \qquad f_e = hg_e,$$

where h is the common of the set, $g_1, g_2, \cdots, g_e$ are relatively prime, and their common is a unit.

29. **Monomials.** Let y be an indeterminate, let a belong to $\mathfrak{R}$ and let m be a non-negative integer. The symbol ay^m is a *monomial* with *coefficient* a and *grade* m. In the presence of $a \neq 0$, the monomial has *degree* m. If $a = 0$, the monomial has no degree.

30. **Polynomials.**[16] The sum of a finite number of monomials is a *polynomial,* and can be written in the form

$$f = a_0y^m + a_1y^{m-1} + \cdots + a_{m-1}y + a_m, \tag{30.1}$$

where some of the a's may be zero. The *grade* of the monomial is m. a_0 is called the *initial* of f. In the presence of $a_0 \neq 0$, the *degree* of f is m. If every a is zero, the polynomial has no degree. The totality of polynomials whose coefficients are in $\mathfrak{R}$ forms a polynomial ring $\mathfrak{R}[y]$.

The coefficient in the non-zero monomial of highest rank is called the *kernel* of the polynomial. It can be obtained as the last member of the sequence

$$f, f_1, \cdots, f_k,$$

in which each polynomial is the initial of the preceding and the last belongs to $\mathfrak{R}$.

If the coefficients of a polynomial f in $\mathfrak{R}[y]$ are relatively prime, f is a *unit polynomial.* A prime polynomial is necessarily of this type. It is clear that if $f = gh$, any factor of the coefficients of g is a factor of the coefficients of f.

[16] A number of the concepts and terms we shall employ are Ritt's [17]. Initial, rank and leader are essentially the same here as in Ritt's book. We use ordinal instead of class, however, because the latter is employed in Chapter IV in a different sense, which is well established.

Hence if gh is a unit polynomial, both g and h are. The converse of this statement is known as Gauss' theorem. Its proof will be found in [23, I, 74]. Hence we have

THEOREM 30.1. *A product is a unit polynomial if and only if its factors are unit polynomials.*

Any polynomial F can be expressed in the form $F = cf$, where c is a common of its coefficients and f is a unit polynomial, which will be called a *basis* of F.

If a second indeterminate is employed and the symbols of $\mathfrak{R}[y]$ are used for coefficients, a polynomial ring in two indeterminates is obtained. By induction, we reach a polynomial ring $\mathfrak{R}[y_1, \cdots, y_r]$ in r indeterminates. Its *dimension* will be r.

It can be shown [23, I, 73] that if A_7 holds in $\mathfrak{R}$, it holds also in $\mathfrak{R}[y_1, \cdots, y_r]$ and that the units of the polynomial ring are the same as those of $\mathfrak{R}$. Hence a set of polynomials in the ring of dimension r has a common, which may be of degree zero and belong to $\mathfrak{R}$. The term basis will only be used when the polynomial is thought of as being in a single indeterminate.

All the results of §28 apply if the $\mathfrak{R}$ of that section is regarded as $\mathfrak{R}[y_1, \cdots, y_r]$.

As is well known, the differential coefficients in $\mathfrak{R}[y_1, \cdots, y_r]$ can be defined algebraically in the same way as the derivatives of a Grassmann polynomial (§10). We shall assume that the ordinary rules for differentiating polynomials have been established. It might be remarked, however, that if the algebra of commutative rings were being developed on a postulational basis, differentiation could be introduced for $\mathfrak{R}[y_1, \cdots, y_r]$ by postulating that $\mathfrak{R}$ is the ring of constants, the ring $\mathfrak{R}$ involved in assumption D_1 is $\mathfrak{R}[y_1, \cdots, y_r]$, and the y's are the variables. The differential coefficient is then the usual one and in particular $(\mathfrak{R}[y_1, \cdots, y_r])' = \mathfrak{R}[y_1, \cdots, y_r]$.

Consider the relation (28.3) in which f is interpreted as belonging to $\mathfrak{R}[y]$. Differentiation with respect to y gives

$$(30.2)\quad f' = Af_1^{d_1-1}f_2^{d_2-1}\cdots f_s^{d_s-1}(d_1f_1'f_2\cdots f_s + d_2f_1f_2'\cdots f_s + \cdots + d_sf_1f_2\cdots f_s').$$

Any common of f and f' is an associate of

$$(30.3)\qquad Af_1^{d_1-1}f_2^{d_2-1}\cdots f_s^{d_s-1},$$

for no f_i is a factor of the parenthesis on the right of (30.2).

For many purposes, the indeterminates must be considered to be in a definite order, which will be assumed that of *increasing subscript*. This ordering can be effected by the first cote (5.6).

Let f be written as a polynomial in the single indeterminate y_α with polynomials in all the other indeterminates for coefficients. The grade ρ_α of this polynomial is called the *grade of f in y_α*.

The symbol $\rho = (\rho_r, \rho_{r-1}, \cdots, \rho_1)$ is called the *rank* of f.

Any polynomial of rank $(0, 0, \cdots, 0)$ belongs to $\mathfrak{R}$. If it belongs to $\mathfrak{R}$, it is said to have *ordinal* zero.

If the rank of f is $(0, \cdots, 0, \rho_k, \cdots, \rho_1)$, $\rho_k \neq 0$, k is called the *ordinal*[17] of f and y_k its *leader*. By the *initial* of f is meant its initial when written as a polynomial in its leader as single indeterminate. The initial is of lower ordinal than f.

THEOREM 30.2. *The degree of a non-zero product is the sum of the degrees of its factors.*

31. **Resultants.** Numerous modifications of the usual method of treating this subject are essential for the applications we shall make, so we develop the matter from the beginning.

Let

$$\begin{aligned} f(y) &= a_0 y^m + a_1 y^{m-1} + \cdots + a_{m-1} y + a_m \qquad (a_0 \neq 0), \\ g(y) &= b_0 y^n + b_1 y^{n-1} + \cdots + b_{n-1} y + b_n \qquad (b_0 \neq 0) \end{aligned} \tag{31.1}$$

be two members of $\mathfrak{R}[y]$. Suppose for convenience $m \geqq n$. Let p be a non-negative integer satisfying $l = m + n - 2p > 0$. Multiply f and g by y^{n-p-1}, $y^{n-p-2}, \cdots, y, 1$ and $1, y, \cdots, y^{m-p-2}, y^{m-p-1}$, respectively, to obtain the system

$$\begin{array}{rl} y^{n-p-1} f = & a_0 y^{m+n-p-1} + a_1 y^{m+n-p-2} + \cdots + a_m y^{n-p-1}, \\ y^{n-p-2} f = & \qquad a_0 y^{m+n-p-2} + \cdots + a_m y^{n-p-2}, \\ \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f = & \qquad\qquad a_0 y^m + \cdots + a_m \\ g = & \qquad\qquad b_0 y^n + \cdots + b_n \\ \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y^{m-p-2} g = & \qquad b_0 y^{m+n-p-2} + \cdots + b_n y^{m-p-2}, \\ y^{m-p-1} g = & b_0 y^{m+n-p-1} + b_1 y^{m+n-p-2} + \cdots + b_n y^{m-p-1}, \end{array} \tag{31.2}$$

it being agreed that a multiplication is omitted if an exponent of y becomes negative in it.

Regard (31.2) as a linear system in the $m + n - p$ variables $y^{m+n-p-1}, \cdots, 1$ and call its matrix M_p. The determinant of order l containing the first l columns of M_p is called the pth *resultant* of f and g and is denoted by R_p, or by $R_p(f, g)$, when putting f and g in evidence is desirable.

$R = R_0$ is called the *resultant*. R_p can be obtained from R_{p-1} by omitting its first and last rows and its first and last columns.

It is clear that every $R_p = 0$ if $a_0 = b_0 = 0$. Hence the non-vanishing of any R_p is a sufficient condition that one of the initials be not zero.

If $m > n$, we readily find

$$R_n = (-1)^{m-n-1} b_0^{m-n} \neq 0. \tag{31.3}$$

[17] It seems convenient to let r be the highest ordinal and 1 the lowest, that is, the ordinal of a polynomial is that of its leader in accordance with (5.6).

If $m = n$, the last of the R's is

$$R_{m-1} = a_0 b_1 - a_1 b_0, \tag{31.4}$$

and R_m is not yet defined. We define it as 1.

Let h be a third polynomial

$$h = c_0 y^p + c_1 y^{p-1} + \cdots + c_p,$$

whose coefficients do not necessarily belong to $\mathfrak{R}$ but can be subjected to addition and multiplication as if they belonged to $\mathfrak{R}$. We wish to form the resultant of fh and gh. The coefficients of these products are most readily found by the method of multiplication with detached coefficients. When the resultant of fh, gh has been written out, it can be transformed [2, 55] by elementary transformations of its columns into the resultant of $y^p f$, $y^p g$ with every element multiplied by c_0. For the first column is $c_0 a_0, 0, \cdots, 0, c_0 b_0$ at the start and becomes $a_0, 0, \cdots, 0, b_0$ after division by c_0. If the new first column is multiplied by c_1 and subtracted from the second, the latter becomes $(c_0 a_1, c_0 a_0, 0, \cdots, 0, c_0 b_0, c_0 b_1)$, or after division by c_0, $(a_1, a_0, 0, \cdots, 0, b_0, b_1)$. Continuing, we get the result as stated.

It is clear that the last p columns of $R(fh, gh)$ when transformed consist of zeros. Hence, p being the grade of h,

$$R_i(fh, gh) = 0 \qquad (i = 0, 1, \cdots, p - 1). \tag{31.5}$$

The above method of forming $R(fh, gh)$ clearly holds for all the resultants, so that we have

$$R_{p+q}(fh, gh) = c_0^{m+n-2q} R_q(f, g). \tag{31.6}$$

It is convenient to state (31.5) as

THEOREM 31.1. *If two polynomials have a factor of degree q in common, their first q resultants are zero.*

32. **Determination of a common.** The polynomials f and g being given, let us try to satisfy the equations

$$f = hf_1, \qquad g = -hg_1 \tag{32.1}$$

by polynomials f_1, g_1, h, whose degrees, although not specified, are subjected to the condition that the degree of h is at least one.

A necessary condition is clearly

$$fg_1 + f_1 g = 0, \qquad f_1 g_1 \neq 0. \tag{32.2}$$

Conversely, if f_1, g_1 satisfy (32.2), and have degrees not exceeding $m - p$, $n - p$, respectively, prime factors of g with total degree at least p must be factors of f because the prime factors of g must all be factors of fg_1 by Theorem 28.1 and the total degree of those factors which are contained in g_1 cannot exceed $n - p$. Hence there is a polynomial of degree $p > 0$ satisfying (32.1).

Let us assume for f_1, g_1 polynomials of grades $m - 1$, $n - 1$ and put

$$(32.3)\qquad \begin{aligned} f_1 &= x_{m+n-1}y^{m-1} + x_{m+n-2}y^{m-2} + \cdots + x_n, \\ g_1 &= x_0 y^{n-1} + x_1 y^{n-2} + \cdots + x_{n-1}. \end{aligned}$$

The result of substituting in (32.2) and equating to zero coefficients of the powers of y is a system of $m + n$ linear homogeneous equations (§16) in the $m + n$ x's. The determinant of this system is the transpose (i.e., the result of interchanging rows and columns) of the resultant $R(f, g)$. Theorem 16.1 gives the condition for the existence in $\mathfrak{R}[y]$ of polynomials satisfying (32.2) and hence the existence of polynomials satisfying (32.1). (The degree of h may necessarily be greater than one because the initials of f_1, g_1 may have to be zero.) We state this result as

THEOREM 32.1. *Two polynomials with non-zero initials in $\mathfrak{R}[y]$ have in common a factor which is in $\mathfrak{R}[y]$ but not in $\mathfrak{R}$ if and only if their resultant is zero.*

From this it follows that for the factors of (28.3)

$$(32.4)\qquad R(f_i, f_j) \neq 0.$$

If we interpret h as the common of f and g, by Theorem 32.1 we have $R(f_1, g_1) \neq 0$. Making $q = 0$ and replacing f, g by f_1, g_1 in (31.6) we infer $R_p(f, g) \neq 0$. On the other hand, (31.5) shows that $R_i(f, g) = 0$ ($i = 0, \cdots, p - 1$). Hence we have

$$(32.5)\qquad R = R_1 = \cdots = R_{p-1} = 0, \qquad R_p \neq 0,$$

or, in words,

THEOREM 32.2. *The degree of a common of two polynomials is the same as the index of the first non-vanishing resultant.*

The case $p = n$ merits special attention.

If $m > n = p$, g_1 belongs to $\mathfrak{R}$. From (32.2), g_1 must be a factor of all the coefficients of g. Hence, *f has any basis of g as a factor.*

If $m = n = p$, both f_1 and g_1 belong to $\mathfrak{R}$. From (32.2) f_1 is a factor of the coefficients of f and g_1 a factor of those of g. Hence *f and g have bases which are associates.* In particular, *if f and g are unit polynomials, f and g are associates.*

To determine a common, rewrite (31.2) in the form

$$(32.6)\qquad K^i = a^i_j y^{m+n-p-j} + L^i \qquad (i = 1, \cdots, l; j = 1, \cdots, l - 1),$$

where a superscript denotes a power only if it is on y. Multiplication of (32.6) by $\mathfrak{A}^l_i$ (see §11) gives

$$(32.7)\qquad Pf + Qg = R_p[L, l] = \varphi,$$

where P, Q are in $\mathfrak{R}[y]$ and the notation is that of §19.

The right member of (32.7) is a determinant, which is easily found from R_p by replacing its last column by the L's. We also get φ if we make the substitution

$$L^i = y^{n-p-i}f \qquad (i = 1, \cdots, n-p),$$
$$L^i = y^{i-n+p-1}g \qquad (i = n-p+1, \cdots, l),$$

instead of employing the actual values of the L's in $R_p[L, l]$.

If h is a common of f and g, it is clear from (32.7) that h is a factor of the right member, and since the degree of φ is p, we have

$$\varphi = Ah,$$

where A belongs to $\mathfrak{R}$. To find h, therefore, we find a basis of φ and multiply it by a common of all the coefficients of f and g. In particular, if f and g are unit polynomials, any basis of φ is a common for f and g.

The *quotients* f_1, g_1 of f, g by the common h are readily determined ($l > 0$). Let

$$\begin{aligned} f_1 &= x_{l+1}y^{m-p} + x_l y^{m-p-1} + \cdots + x_{n-p+1}, \\ g_1 &= x_0 y^{n-p} + x_1 y^{n-p-1} + \cdots + x_{n-p}. \end{aligned} \tag{32.8}$$

The initials x_{l+1}, x_0 are a_0, $-b_0$ divided by the initial of h. Substituting from (32.8) in (32.2) and equating coefficients to zero, we obtain a system of l nonhomogeneous equations in the l variables $x_1, \cdots, x_l$, whose determinant is the transpose of R_p. This system is readily solved (see (16.4)) in the form

$$R_p x_i = -x_0 R_p[i, a] - x_{l+1} R_p[i, b] \qquad (i = 1, \cdots, l), \tag{32.9}$$

where a_i, b_i are to be interpreted as zero for $i > m$, n, respectively. The right members, of course, have R_p as a factor because the x's are known to be in $\mathfrak{R}$.

If the ring $\mathfrak{R}$ in the preceding discussion is replaced by $\mathfrak{R}[y_1, \cdots, y_{r-1}]$ and the indeterminate y adjoined is y_r, the above discussion is applicable for the determination of the common of two members of $\mathfrak{R}[y_1, \cdots, y_r]$. Conditions (32.5) are to be interpreted as follows: $R, R_1, \cdots, R_{p-1}$ belong to $\mathfrak{O}$, whereas some coefficient of R_p belongs to $\mathfrak{R}$.

In the application made in Chapter VI, $y_1, \cdots, y_{r-1}$ are to be identified with symbols of $\mathfrak{R}_C$. With this interpretation $\mathfrak{R}[y_1, \cdots, y_{r-1}]$ belongs to $\mathfrak{R}_C$ and conditions (32.5) mean that $R, R_1, \cdots, R_{p-1}$ belong to $\mathfrak{O}_C$ and R_p belongs to $\mathfrak{R}_C$ when this identification is made. We cannot conclude that the right member of (32.9) has R_p as a factor in $\mathfrak{R}[y_1, \cdots, y_r]$. We do have, however, that the right members of (32.9) are formed by ring operations in $\mathfrak{R}[y_1, \cdots, y_r]$. Let F_1 be the member of $\mathfrak{R}[y_1, \cdots, y_r]$ of grade $m - p$ in y_r with initial $a_0 R_p$ and its other coefficients computed from (32.9), and let G_1 be similarly defined. Then we have, φ being defined by (32.7), as equations in $\mathfrak{R}_C[y_r]$

$$R_p^2 f = \varphi F_1, \qquad R_p^2 g = -\varphi G_1. \tag{32.10}$$

In this case also, φ will be called a common of f and g.

33. **Discriminants.** If g is the derivative f' of f in the above discussion, $a_0^{-1}R_i$ is called the ith *discriminant*[18] of f and is denoted by D_i or $D_i(f)$. If

$$D = D_1 = \cdots = D_{p-1} = 0, \qquad D_p \neq 0, \tag{33.1}$$

we may determine by ring operations the quotient of f by a common of f and f'. Let it be Ψ. Then (§30) we may take

$$\Psi = f_1 f_2 \cdots f_s, \tag{33.2}$$

whence

$$\Psi' = f_1' f_2 \cdots f_s + f_1 f_2' \cdots f_s + \cdots + f_1 f_2 \cdots f_s'.$$

From this it is clear that Ψ and Ψ' are relatively prime and that the same is true of every pair f_i and f_i'. Hence

$$D(\Psi) \neq 0, \qquad D(f_i) \neq 0, \qquad R(f_i, f_j) \neq 0. \tag{33.3}$$

We readily find

THEOREM 33.1. *The multiplicity of some prime factor of a polynomial with non-zero initials exceeds one if and only if the discriminant is zero.*

34. **The zeros of a polynomial.** Let α and the coefficients of f belong to $\mathfrak{R}$. By performing the multiplications and arranging as a polynomial in $(y - \alpha)$ we have

$$\begin{aligned} f(y) = f[(y - \alpha) + \alpha] = b_0(y - \alpha)^m + b_1(y - \alpha)^{m-1} + \cdots \\ + b_{m-1}(y - \alpha) + b_m, \end{aligned} \tag{34.1}$$

where the b's are formed from the a's and α by addition and multiplication alone and therefore belong to $\mathfrak{R}$. By the substitution $y = \alpha$ we find (see Chapter II for the notation) $f(\alpha) = b_m$. Hence (34.1) can be rewritten

$$f(y) = (y - \alpha)\varphi(y) + f(\alpha), \tag{34.2}$$

where φ belongs to $\mathfrak{R}[y]$. The coefficient of y^{m-1} in φ is a_0. Hence if $a_0 \neq 0$, φ has the same initial as f and degree one less.

If α is a zero of f, we have

$$f(y) = (y - \alpha)\varphi(y). \tag{34.3}$$

Since the converse is true, we have the

FACTOR THEOREM 34.1. *A polynomial in $\mathfrak{R}[y]$ has the linear factor $y - \alpha$ in $\mathfrak{R}[y]$ if and only if it has the zero α.*

It can be shown that every integrity domain can be imbedded in an $\mathfrak{R}_C$ such that every polynomial in $\mathfrak{R}_C[y]$ has a zero in $\mathfrak{R}_C[y]$ [23, I, 199]. We shall take

[18] It is clear that the first column of $R_i(f, f')$ is divisible by a_0 and that the discriminant is consequently a polynomial in the coefficients.

this result as an assumption, which is not independent of those at the basis of our algebra and hence need not be explicitly formulated. Familiar reasoning gives

THEOREM 34.2. *Every polynomial in* $\mathfrak{R}[y]$ *can be given in a closed* $\mathfrak{R}_c[y]$ *containing* $\mathfrak{R}[y]$ *the form*

$$f = a_0(y - \alpha_1) \cdots (y - \alpha_m) \qquad (a_0 \neq 0), \tag{34.4}$$

where the α*'s belong to* $\mathfrak{R}_c$, *and* m *is the degree of* f. *There is no integrity domain containing* $\mathfrak{R}_c$ *in which* f *has zeros other than the* m α*'s.*

Every polynomial having a number of zeros greater than the exponent of the highest power of y *appearing in its formal expression belongs to* $\mathfrak{O}$.

By taking $h = y - \alpha$ in (31.5) we see that $R(f, g) = 0$ is a necessary condition for f and g to have a zero in $\mathfrak{R}_c$. By Theorem 32.2 the degree of the common of f and g is at least one. By Theorem 34.2 the common has p zeros in $\mathfrak{R}_c$. Hence we have

THEOREM 34.3. *Two polynomials with non-zero initials have a zero in common if and only if their resultant is zero. If the common of the polynomials is of degree* p, *they have exactly* p *zeros in common.*

THEOREM 34.4. *If* f *and* g *have a zero in* $\mathfrak{R}_c$, *they have a common factor which is in* $\mathfrak{R}[y]$ *but not in* $\mathfrak{R}$.

Theorem 34.3 applied to f and f' shows that they have a zero in common if and only if $D(f)$ is zero. Hence the zeros of (33.2) are all distinct, and in particular we have

THEOREM 34.5. *The zeros of a prime polynomial are all distinct.*

A unit polynomial is called *simple* if its zeros are distinct. In accordance with Theorem 34.2 a simple polynomial has exactly m zeros if its degree is m. Its non-zeros are obtained by excluding these m from $\mathfrak{R}_c$.

THEOREM 34.6. *Two simple polynomials have the same zeros if and only if they are associates. This is also the necessary and sufficient condition that they have the same non-zeros.*

To prove the necessity of the first condition, we remark that the degrees must be equal. Hence we have $m = n$, $R = \cdots = R_{m-1} = 0$, and the result follows from the second remark following Theorem 32.2.

To prove the necessity of the second condition, we remark that the polynomials must have the same zeros and the result follows from the first part.

The sufficiency is obvious in both cases.

The following useful result is readily proved.

THEOREM 34.7. *Not all the roots of a simple inequation can satisfy an inequation of equal ordinal and higher degree.*

35. **Exclusion of finite rings.** If $\mathfrak{R}$ contains only a finite number of distinct symbols, the zeros of a polynomial f may be the same as the non-zeros of another polynomial g. This happens, for example, if $\mathfrak{R}$ is the residue system [23, I, 55] modulo 5 and

$$f = y(y - 1), \qquad g = (y - 2)(y - 3)(y - 4).$$

To obviate this difficulty, we shall occasionally need the assumption

A_8. $\mathfrak{R}$ *contains the rational integers.*

This means that the set of identities β mentioned in footnote 6, page 10 is vacuous.

THEOREM 35.1. *Every polynomial except* 0 *in* $\mathfrak{R}[y]$ *has an infinite number of non-zeros.*

36. **Sets of unit monomials.** In accordance with the definition (§30) of unit monomial, the coefficient of such a monomial is a unit, which we shall take as 1 for convenience. Let M be a finite set of monomials $x_1^{i_1} \cdots x_n^{i_n}$. If for k fixed the maximum value of i_k is j_k, the symbol $j_1 \cdots j_n$ will be called the *index* of M. If M contains a single monomial, the index is the symbol $i_1 \cdots i_n$ and completely defines the monomial.

The number of monomials with a given index is finite: if $i_1 \cdots i_n$ belongs to M of index $j_1 \cdots j_n$, then $i_k \leqq j_k$.

With each monomial m of M associate certain of the variables x which will be called *multipliers*, according to the following rule: *a variable* x_k *is multiplier for* $m = i_1 \cdots i_n$ *if* $i_k = j_k$, *where* $j_1 \cdots j_n$ *is the index; a variable is non-multiplier if* $i_k < j_k$.

If the product of a monomial m of M by one of its non-multipliers x is not in M, we form the product $m^1 = mx$ and adjoin it to M to give the set M^1. The index of M^1 is the same as that of M: if x is non-multiplier for m, its exponent in m is less than the maximum for M and in m^1 consequently does not exceed that maximum. For the same reason, the multipliers of the monomials of M are the same in M^1 as in M. Moreover, all the multipliers of m are multipliers of m^1, and if the degree of m^1 in x attains the maximum, m^1 has x for multiplier in addition.

Let the above process be repeated on M^1 to give a sequence $M^0 = M, M^1, \ldots$. Since the number of monomials in a set of given index is finite, there exists a k such that $M^k = M^{k+1}$. The M^k corresponding to the least k with this property is called the *complete set determined by* M, and is denoted by M^*.

M^* contains a monomial m^* whose index is that of M and which is the least common multiple of the monomials in M. All of the variables x are multipliers for m^*, and m^* is the only member of M^* with this property. M^* consists of all monomials dividing m^* and divisible by at least one monomial of M.

If m and m' are two monomials of M^* related by the formula $m = \lambda m'$, λ contains no multiplier of m'; for if x is multiplier for m', its exponent is maximum in m' so that λ cannot contain x. If $m = i_1 \cdots i_n$ and $m' = j_1 \cdots j_n$ are

related by $m = m'x$, where x is non-multiplier for m', all of the differences $i_1 - j_1, \cdots, i_n - j_n$ are zero except one, which is positive. m is said to be of *higher absolute rank* than m' if $m \neq m'$ and none of these differences is negative, as is the case here. (This agrees with the rank defined on page 49.)

A fundamental property of the complete set is

THEOREM 36.1. *Every multiple of a monomial of a set is the product of a unique monomial of the corresponding complete set by multipliers alone. This monomial is called the generator of all its multiples. The generator of a given monomial is characterized as the monomial in the complete set of highest absolute rank which divides it.*

The complete set accordingly gives an elegant way of describing the set of all multiples of a given finite set M. It is also desirable to have a similar way of handling the non-multiples. To this end, we form the set $\bar{M}$ containing all divisors of the monomials of M^* which are not contained in M. The monomials of $\bar{M}$ are then given the multipliers and non-multipliers which they would have if they were members of M. The set $\bar{M}$ is called the *complementary* set and has the properties given in

THEOREM 36.2. *Every multiple of a monomial of $\bar{M}$ by a non-multiplier is contained in $\bar{M}$ or M^*.*

A monomial which is a non-multiple of every monomial of M is the product of a unique monomial m of $\bar{M}$ by multipliers alone. m is again called the generator and is the monomial of highest absolute rank in $\bar{M}$ dividing the given monomial.

Given a set M, the easiest way to construct M^* and $\bar{M}$ is the following. Find the least common multiple m^*. Write down all its factors. The result is the set $M^* + \bar{M}$. A monomial belongs to M^* or $\bar{M}$ according as it is a multiple of some monomial in M or of none.

Let there be given r sets $M_\alpha (\alpha = 1, 2, \cdots, r)$ of monomials. With each monomial m of $\bar{M}_\alpha$ associate an arbitrary function $I_{\alpha m}$ of its multipliers. The totality of functions $I_{\alpha m}$ will be called an *initial determination* I.

37. **Relative complete sets.** The complete set and initial determination defined in the preceding section are the most satisfactory for theoretical purposes. In particular, they do not depend on the ordering of the independent variables x. Because of this property, they may be called *absolute*.

They have, however, certain redundancies, which can be avoided in practice by employing a *relative complete set,* now to be defined.

Let the monomials of the (absolute) complete set M^* be put into mutually exclusive subsets M_m, one of which corresponds to each monomial m of the original set M, in accordance with the following rule. A monomial of M^* is placed in M_m if and only if m is the monomial of highest rank (§§5 and 30) in M of which it is a multiple. Certain omissions are then to be performed. If m' and m'' are two monomials of M_m, if m'' is not in M and if the quotient of m'' by m' involves only multipliers of m'', m'' is omitted and m' is given the

multipliers of m'' in addition to those it already possesses. The order in which omissions are performed is that of increasing relative rank. After all possible eliminations of this kind have been performed, the totality of monomials remaining in the sets M_m is called the *relative complete set* arising from M for the given order of the variables. A similar reduction is applied to $\bar{M}$ to give the *relative complementary set*. There is no division into sets M_m, of course. The condition for the omission of m'' is: the quotient of m'' by m' involves only multipliers of m'' and every multiplier of m' is a multiplier of m''; or the quotient of m'' by m' involves only multipliers of m'. The *relative initial determination* is obtained by replacing in the definition at the end of §36. $\bar{M}_\alpha$ by the corresponding relative set.

Theorems 36.1 and 36.2 are readily seen to hold if "absolute" is replaced by "relative."

We have also

THEOREM 37.1. *The product of a monomial m by one of its non-multipliers is equal to the product of a unique monomial m' of the complete set by multipliers alone. The monomial m' is of higher rank than m.*

This result, effectively proved in the absolute case, is trivial for that case since the product in question belongs itself to the complete set and needs only to be multiplied by 1.

To prove the theorem in the relative case, we remark that mx, where x is non-multiplier for m, belongs to the absolute complete set. If it is omitted in the formation of the relative set, then $mx = \lambda m'$, where m' is in the relative complete set and is given as multipliers all the variables in λ.

CHAPTER VI

ALGEBRAIC SYSTEMS

In this chapter is developed a new method of solving simultaneous algebraic equations. It is shown that any algebraic system is the product of a finite number of simple systems. For simple systems a positive existence theorem and an equivalence theorem are proved. The resolution into simple factors is not unique, but a test for equivalence of general systems can be based upon it. Finally we prove the existence of a resolution into prime systems which is essentially unique.

38. **Generalities.** The polynomials of $\mathfrak{R}[y_1, \cdots, y_r]$ are single-valued functions of the variables y, whose scope will be taken to include every set of r symbols from the closed $\mathfrak{R}_C$. The value of any such function, moreover, belongs to $\mathfrak{R}_C$.

A system whose functions belong to $\mathfrak{R}[y_1, \cdots, y_r]$ is called *algebraic*. Only such systems will be considered and denoted by S in the present chapter.

The *ordinal* and the *minimum ordinal* of S are, respectively, the maximum and minimum ordinal of its polynomials. Let S_α consist of the polynomials in S whose ordinal is α. Any equation belonging to $\mathfrak{O}$ and any inequation belonging to $\mathfrak{R}$ can be suppressed, although they are still implied by any $S_\alpha(\alpha > 0)$. If S_0 is non-vacuous afterwards, S is immediately inconsistent and will be replaced by 1. We put, in any case,

$$\Sigma_k = S_0 + S_1 + \cdots + S_k, \tag{38.1}$$

so that Σ_k consists of the polynomials in S whose ordinal does not exceed k, and in particular $\Sigma_r = S$.

39. **Simple systems.** We shall first show how to find a simple factor (to be defined later) of any algebraic system. We shall use (2.6) constantly and shall be primarily interested in the left factor $S + \bar{f}$, ignoring the other systematically for the present.

Let f be a member of S_k written as a polynomial in its leader y_k with initial a_0 and grade m. Use of (2.6) and an obvious reduction give

$$S = (S + \bar{a}_0)(S - f + [f - a_0 y_k^m] + a_0), \tag{39.1}$$

if f is equation, and an analogous relation if f is inequation. In (39.1) the brackets are, of course, non-removable. Let this process be repeated for every member of ordinal k in the left factor until there is obtained a left factor T which contains as inequation the initial of every one of its members whose ordinal is k.

Let f be a member of T_k with a_0, a_1 for first two coefficients when written as a polynomial in its leader. Let a_0, a_1 be written as polynomials in the

indeterminate of highest ordinal in either. Let $R(a_0, a_1)$, $R_1, \cdots, R_{p-1}$ belong to $\mathfrak{D}$ or be obviously implied as equations by S, and let R_p not belong to $\mathfrak{D}$. On the assumption $R_p \neq 0$, determine a common of a_0 and a_1. Continuing, we obtain a common h of all the coefficients of f in the presence of $A \neq 0$, where A is the product of certain resultants. This leads to a factorization

$$(39.2) \qquad T = (T + \bar{A} - f + f/h)(T + A),$$

since $\bar{a}_0$ implies $\bar{h}$ (f is to be barred, if it is inequation). Repetition gives a left factor U such that the common of the coefficients of every member in U_k belongs to $\mathfrak{R}$.

Let f be a member of U_k. Determine a number q so that the discriminants $D(f), D_1(f), \cdots, D_{q-1}(f)$ belong to $\mathfrak{D}$ or are obviously implied as equations by S, whereas $D_q(f)$ does not belong to $\mathfrak{D}$ (although its vanishing may ultimately be implied by S). By ring operations in $\mathfrak{R}[y_1, \cdots, y_k]$ we determine polynomials such that

$$D_q^2 f = Ff_1, \qquad D_q^2 f' = Ff_2, \qquad D(f_1) \neq 0,$$

for any values of $y_1, \cdots, y_{k-1}$ contained in a root of $U + \bar{D}_q$. We consequently get the factorization

$$(39.3) \qquad U = (U + \bar{D}_q - f + f_1)(U + D_q),$$

if f is equation, and if f is inequation, we need only bar f, f_1 to have the appropriate result. Let this be repeated for every member of ordinal k in the left factor until there is obtained a left factor V which implies as inequations the initial and discriminant of every one of its members whose ordinal is k.

Let f be an equation and g a polynomial of V_k. Let the resultants $R(f, g)$, $R_1(f, g), \cdots, R_{p-1}(f, g)$ belong to $\mathfrak{D}$ or obviously be implied as equations by S, whereas $R_p(f, g)$ does not belong to $\mathfrak{D}$. By ring operations in $\mathfrak{R}[y_1, \cdots, y_k]$ we determine polynomials in y_k such that

$$R_p^2 f = \varphi f_1, \qquad R_p^2 = \varphi g_1,$$

and f_1, g_1, φ are relatively prime, for any values of $y_1, \cdots, y_{k-1}$ contained in a root of $V + \bar{R}_p$. There are two cases: (i) $p = 0$, $f + g = 1$, $f + \bar{g} = f$; (ii) $p > 0$, $f + g = \varphi$, $f + \bar{g} = f_1$. In every case, therefore, the polynomial g can be eliminated from the left factor of

$$(39.4) \qquad V = (V + \bar{R}_p)(V + R_p).$$

If V_k contains an equation, by repetition of the above we can obtain a left factor which has only one polynomial (an equation) of ordinal k.

If V_k contains only inequations, we multiply all of them together and replace the product by an inequation whose discriminant is implied as an inequation by the left factor, just as was done in (39.3).

We begin with $k = r$ and determine a left factor W by the above process. Then make $k = r - 1$ and apply the process to W. Continuing, we finally get a left factor s of S with the following properties:

(i) *Every polynomial in* s_α *is simple* (§34) *when* $y_1, \cdots, y_{\alpha-1}$ *constitute a root of* $s_1 + s_2 + \cdots + s_{\alpha-1}$.

(ii) *Each* s_α *contains at most one polynomial.*

(iii) *Each* $s_1 + \cdots + s_{\alpha-1}$ *implies as inequations the initial and discriminant of* s_α *and if* $s_1 + \cdots + s_{\alpha-1}$ *is vacuous, the initial and discriminant of* s_α *belong to* $\mathfrak{R}$.

(iv) *The common of the coefficients of any member of s written as a polynomial in its leader belongs to* $\mathfrak{R}$.

Only (iii) needs any demonstration. This is accomplished by remarking that in the above process the initial and discriminant of s_α are placed in $s_1 + \cdots + s_{\alpha-1}$ and later are only replaced by polynomials which imply them.

Any of the factors, other than left factors, which have been ignored above, are readily seen to arise from the original S by applying a reduction algorithm (§4). Hence after a finite number of steps the original S is replaced by a product of factors all having properties (i)–(iv).

In a system s satisfying (i)–(iii) y_α is a *conditioned* or *unconditioned* indeterminate according as s_α is non-vacuous or vacuous. If s_α is an equation, y_α is called *principal*; otherwise y_α is *parametric*. The degree of the equation in which the principal variable y_α is leader is called the degree of s_α and also the *degree of s in* y_α, for a reason which will be apparent from the reduction process now to be developed.

Let f be an equation of ordinal α, initial a_0, and degree τ from s. Let a polynomial G of higher ordinal be written as a polynomial in $y_{\alpha+1}, \cdots, y_r$. Let the coefficient of the monomial M in G be g, and let it be written as a polynomial of grade σ in y_α with initial b_0. If $\sigma \geqq \tau$, we have that

$$h = a_0 G - b_0 f y_\alpha^{\sigma-\tau} M \tag{39.5}$$

is a polynomial in which the coefficient of M is of grade less than σ. Also, $s = s - G + h$ because s implies $\bar{a}_0$. A repetition of the process will finally replace G by a polynomial in which the coefficient of M is of lower grade than τ in y_α. By repetition we get a system with the following property.

(v) *The degree of any equation of s in a principal indeterminate exceeds the grade of any other polynomial G of the system in that indeterminate.*

The new system may not have properties (i)–(iv). If not, we recommence on it with (39.1), (39.2), (39.3), (39.4). After a finite number of steps we express the original S as a product of factors having properties (i)–(v).

A system s having properties (i)–(v) is called a *simple system.* If the simple system arises by applying the above process to a system S, it is called a *canonical factor* of S. That S can have as a factor a simple system which is not a canonical factor is seen from the following example:

$$S = \bar{x}, \qquad T = x - 1, \qquad U = \overline{x(x-1)}, \qquad S = TU.$$

Clearly we have the following

THEOREM 39.1. *Every algebraic system from $\mathfrak{R}[y_1, \cdots, y_r]$ can be expressed by a finite number of ring operations in $\mathfrak{R}[y_1, \cdots, y_r]$ as the product of a finite number of canonical factors, no two of which have a root in common.*

The process of (39.5) can be applied, of course, to any polynomial G, whether G belongs to a simple system S or not. If applied to each polynomial of a system T, there results a system T^*, every polynomial of which has grade in each principal indeterminate of S less than the degree of S in that indeterminate. T^* will be called *reduced*[19] *with respect to* S.

THEOREM 39.2. *If the simple system $S \geqq T$, then $S \geqq T^*$ and conversely.*

40. **Existence theorem for simple systems.** We shall now prove the following positive existence theorem.

THEOREM 40.1. *A simple system S of positive minimum ordinal is consistent and any root of Σ_α is contained in a root of $\Sigma_{\alpha+1}$.*

Let $f_{l+1}, \cdots, f_r (l \geqq 0)$ be polynomials of a simple system S, the notation having been adjusted so that f_i is of ordinal i and degree d_i. The unconditioned indeterminates are then $y_1, \cdots, y_l$. Since the initial and discriminant of f_{l+1} belong to $\mathfrak{R}$, if the unconditioned indeterminates are replaced by any symbols of $\mathfrak{R}_C$, f_{l+1} becomes a polynomial of degree d_{l+1} in the single indeterminate y_{l+1} whose discriminant is not zero. Hence by Theorems 33.1 and 34.2 f_{l+1} has d_{l+1} distinct zeros. If f_{l+1} is equation, we take any one of these for y_{l+1}. If f_{l+1} is inequation, we take any symbol of $\mathfrak{R}_C$ except those d_{l+1} for y_{l+1}.

Similarly, the initial and discriminant of f_{l+2} do not vanish for the values already assigned $y_1, \cdots, y_{l+1}$. It accordingly becomes a polynomial of degree d_{l+2} in the unknown y_{l+2} and has d_{l+2} distinct zeros.

Proceeding in the way which should now be obvious, we construct a root of S, and the theorem is proved.

If $y_1, \cdots, y_l$ are regarded as variables whose scope is $\mathfrak{R}_C$, the conditioned indeterminates are functions of them.

The foregoing theorem gives

THEOREM 40.2. *An algebraic system is consistent if and only if one of its simple factors is different from* 1.

We shall next prove

THEOREM 40.3. *Any algebraic system comprising a single polynomial with at least one coefficient from $\mathfrak{R}$ is consistent. An algebraic system containing only inequations is consistent if every inequation has at least one coefficient from $\mathfrak{R}$.*

Consider the kernel of the polynomial. It is merely a matter of notation to assume that it is $ay_s^{e_s} y_{s+1}^{e_{s+1}} \cdots y_r^{e_r}$. Forming the sequence $f = f_r, f_{r-1}, f_{r-2}, \cdots,$

[19] This phrase is employed in essentially the same sense by Ritt [17, 6].

in which each member is the initial of the preceding, we end up with the a from $\mathfrak{R}$. By using Theorem 34.2 and the method of proof of Theorem 40.1 we get the desired result.

If $y_i^{e_i}$ is the highest power of y_i which appears in a system S, $e_1 e_2 \cdots e_r$ is called the *index* of S.

THEOREM 40.4. *Let y_i denote a principal indeterminate of a consistent simple system S. If S implies an equation f whose grade in every y_i is less than the degree of S in that y_i, then f is zero.*

Let the polynomial f be written as a polynomial in y_r. When any root of Σ_{r-1} is substituted in its coefficients, it is satisfied by at least d_r roots. Consequently its coefficients are zero for any solution of Σ_{r-1} by Theorem 34.2. Writing the coefficients as polynomials in y_{r-1} and continuing the argument, we finally get a set of polynomials in the unconditioned indeterminates. Since every set of values for those indeterminates gives a zero, all the coefficients, that is, all the coefficients (from $\mathfrak{R}$) of f, must be zero, and the result is established.

THEOREM 40.5. *If the simple S implies a simple system T, every equation of T becomes zero when reduced with respect to S and every inequation f_α of S which corresponds to an inequation g_α of T becomes zero when reduced with respect to $S + g_\alpha$. If S_α is vacuous, so also is T_α.*

The result follows immediately from Theorem 40.4 in the case of an equation in T. Consider the system $S' = S - \bar{f}_\alpha + g_\alpha$. Since $S_{\alpha-1}$ implies $T_{\alpha-1}$, S'_α is simple and consistent. Any root of S'_α for which $f_\alpha \neq 0$ could be imbedded in a root of S. This would involve a contradiction. Hence S'_α implies f_α and the second case follows from the first. This completes the proof.

That S actually implies $\bar{g}_\alpha$ follows from

$$S + g_\alpha = S - \bar{f}_\alpha + g_\alpha + \bar{f}_\alpha = S - \bar{f}_\alpha + g_\alpha + \bar{f}_\alpha + f_\alpha = 1.$$

We have also as an obvious corollary

THEOREM 40.6. *A simple system S implies an equation f if and only if f becomes zero when reduced with respect to S.*

41. **Equivalence of simple systems.** We shall now prove

THEOREM 41.1. *Two consistent simple systems with coefficients in an infinite $\mathfrak{R}$ are equivalent if and only if their polynomials can be put in a one-to-one correspondence such that corresponding polynomials f, g with initials a_0, b_0 have the same nature (§2) and the polynomials $b_0 f - a_0 g$ become zero when reduced with respect to either system.*

Only the necessity need be proved. The unconditioned indeterminates must be exactly the same for the two systems S, T, since the scope of a conditioned indeterminate in a root of S is a proper subset of $\mathfrak{R}_C$, whereas a root of T can

be found in which any unconditioned indeterminate is any symbol of $\mathfrak{R}_\sigma$. Accordingly we assume that the unconditioned indeterminates are $y_1, \cdots, y_l$ and the polynomials are $f_{l+1}, \cdots, f_r, g_{l+1}, \cdots, g_r$, each with ordinal equal to its subscript. Now by Theorem 34.6, when any root of $\Sigma_{\alpha-1}$ is substituted in

$$b_{0\alpha} f_\alpha - a_{0\alpha} g_\alpha, \tag{41.1}$$

where $a_{0\alpha}$, $b_{0\alpha}$ represent the initials, the results are zero; that is, (41.1) is implied as an equation by the system. The result then follows by Theorem 40.5.

An example of equivalent simple systems is furnished by

$$f + g + \bar{h} = f_2 + g + \bar{h}, \tag{41.2}$$

where

$$\begin{aligned} f &= (y + x)z + 4xy - 2x^2, \qquad f_2 = (2y + x)z + 5xy - 2x^2, \\ g &= y^2 - xy, \qquad h = x. \end{aligned} \tag{41.3}$$

42. **Equivalence of general systems.** There is no unique factorization theorem for systems in terms of canonical factors, that is, if two equivalent systems are expressed in canonical factors, in general a one-to-one correspondence between factors can not be established so that corresponding factors are equivalent. This is illustrated by the equivalent systems

$$e + f + g + \bar{h} = d + e + f + g + \bar{h}, \tag{42.1}$$

where $d = (z + 1)t$, $e = t$, and f, g, h are given by (41.3). If the order x, y, z, t is adopted for the indeterminates, the left member of (42.1) is already a canonical factor, whereas the right becomes the product of the three systems

$$\begin{array}{lll} \{t, f, g, \overline{x(2x + 1)(x - 1)}\}, & & \\ \{t, z - x - 1, & (1 - 4x)y - x + 1, & 2x^2 - x - 1\}, \\ \{t, z + 1, & (1 - 4x)y + 2x + 1, & 2x^2 - x - 1\}. \end{array}$$

Canonical factors, however, can be made the basis of a test for equivalence in the following way. Let S, T be two equivalent systems, S being expressed in canonical factors thus:

$$S = A_1 A_2 \cdots A_p. \tag{42.2}$$

If f_i are the equations of T and g_j its inequations, necessary and sufficient conditions for equivalence are

$$A_i + f_j = A_i, \tag{42.3}$$

$$A_i + g_k = 1 \tag{42.4}$$

for all values of the indices, together with similar relations in which the rôles of S and T are interchanged. Theorem 40.6 furnishes a ready means of testing whether (42.3) are verified. As for any relation (42.4), we need only find canonical factors for the left member in order to determine whether it is verified.

Let T_1 be a system equivalent to the S in (42.2). If an equation f of ordinal k in a canonical factor B of T_1, when written as a polynomial in its leader, can be expressed as $f = gh$ for all values of $y_1, \cdots, y_{k-1}$ satisfying the polynomials of ordinal not exceeding $k - 1$ in B, we employ the factorization

$$B = (B - f + g)(B - f + h).$$

Because B implies the discriminant of f, the two factors have no root in common. Consequently the first factor implies $\bar{h}$ and the second $\bar{g}$.

We have, therefore, a factorization of T_1 as

$$T_1 = B_{11}B_{12} \cdots B_{1q},$$

where for every pair B_{1j} and B_{1k} there exists a polynomial f such that B_{1j} implies f and B_{1k} implies $\bar{f}$, or vice versa.

For S we accordingly obtain the factorization

$$S = T_1 = \Pi(A_i + B_{1j}), \tag{42.5}$$

where the indices i and j run independently over their respective ranges.

If an equation implied by B_{1j} is not implied by A_i, its adjunction to A_i will either lower the degree of an equation in A_i or replace an inequation or vacuous system in A_i by an equation, when $A_i + B_{1j}$ is reduced to canonical factors. Since neither of these processes can be continued indefinitely, by properly choosing the finite sequence $T_1, T_2, \cdots, T_n$ and factoring the canonical factors of (42.5) by means of T_2, etc., we finally get a factorization

$$S = P_1P_2 \cdots P_t \tag{42.6}$$

such that if

$$T = B_1B_2 \cdots B_q$$

is any system equivalent to S, expressed as was T_1 above, the system P_i implies B_j if $P_i + B_j$ is consistent.

The factors in (42.6) are called *prime*. Two equivalent prime systems are called *associate*.

Clearly we have the following unique factorization theorem, analogous to A_7.

THEOREM 42.1. *Every algebraic system is the product of a finite number of prime systems. Two algebraic systems are equivalent if and only if their prime factors can be put into a one-to-one correspondence such that corresponding factors are associates.*

It is worth noting that *every equation in a prime system is a prime polynomial in its leader for all roots of the subsystem of ordinal not exceeding that of the equation.*

The lack of an algorithm for finding the prime factors of a system makes the notion of prime system less useful in practice than that of simple system. Such an algorithm would in particular provide a means of finding the prime factors of a polynomial.

CHAPTER VII

ALGEBRAIC DIFFERENTIAL SYSTEMS

In this chapter, it is shown that any differential system, whose members are polynomials in the unknowns and their derivatives, can be expressed as the product of a finite number of passive standard systems, whose solution can be made to depend upon the successive solution of a finite number of normal systems. For a normal system an existence theorem is formulated as a postulate.

43. **Generalities.** Let $\mathfrak{R}$ satisfy the assumptions of Chapter IV and let its subring of constants be algebraically closed. Consider an algebraic system S with coefficients from $\mathfrak{R}$. Separate the indeterminates into two sets. Those in the first set will be called *unknowns* and denoted by $z_1, \cdots, z_r$. Those in the second set will be identified with certain of the partial derivatives of the unknowns. It is supposed that the derivatives have been placed in a canonical order.[20]

The ordinal of any indeterminate in S is the ordinal which it has as a derivative in the canonical ordering.

The symbols in $\mathfrak{R}$ are functions of the independent variables x. A symbol of $\mathfrak{R}$ may belong to $\mathfrak{O}$ for some x's and to $\mathfrak{R}$ for others. When we say that a symbol of $\mathfrak{R}$ belongs to $\mathfrak{R}$, however, we shall mean that it belongs to $\mathfrak{R}$ for every value of the x's from their scope.

The system S may imply certain inequations involving the variables x alone. These may be satisfied of themselves or they may be satisfied by modifying the problem and restricting the scope of the x's. If S implies among the x's an equation which is different from zero for some value in the scope, S is to be regarded as inconsistent.

Since S can be factored into simple systems, suppose S is simple. Any derivative of a leader in an equation is called a *principal derivative*. The other derivatives are called *parametric*.

If at least one derivative of an unknown is principal, the unknown is called a *principal unknown*; otherwise, it is *parametric*. In the latter case, the unknown is also a parametric derivative; in the former, it is only a principal derivative if it is itself leader in an equation.

44. **Prolonged systems.** If each equation of a system S is differentiated with respect to each variable x and the results are adjoined to S, the augmented system S' is called the (*first*) *prolonged system of* S. The kth prolonged system S^k is defined by induction.

The derivatives for a solution must of course be computed by means of (23.2),

[20] For most purposes, it is only necessary that the ordering have the properties enumerated in Theorem 5.1.

with y replaced by z. Those formulas amount to a (non-reversible) substitution (24.1) upon the variables and marks of the ring $\mathfrak{R}[x', z']$. A consequence of the calculations of §24 is that under the substitution

(44.1) $$(f^*)' = (f')^*,$$

where the star denotes the result of substituting. The condition for a solution of $f = 0$ is that the substitution convert f into zero, i.e., that $f^* = 0$. Hence from (44.1) $(f')^* = 0$, that is, the substitution converts f' into zero. Hence we have

THEOREM 44.1. *A differential system implies all of its prolonged systems.*

If f is equation in S, we have therefore the equivalence

(44.2) $$S = S + \delta f,$$

where δ is any differential operator. This equality means that the systems are equivalent as *differential* systems. If two differential systems S, T are equivalent as *algebraic* systems, we shall write

(44.3) $$S \approx T$$

when there is need to distinguish between the two kinds of equivalence.

45. **Standard and normal systems.** Let an equation of a simple system S be

$$f = a_0 y^m + \cdots + a_m,$$

and let $\delta = \partial/\partial x$. Then

(45.1) $$\delta f = f'\delta y + \delta a_0 y^m + \delta a_1 y^{m-1} + \cdots + \delta a_m,$$

where f' is the derivative of f with respect to y.

We note the following:

(i) *The leader of δf is δy.*

(ii) *Its initial and discriminant are both f', whose non-vanishing is implied by S.*

The same statements are true if δ is any differential symbol.

We now describe two important modifications of S which can be made by means of its prolonged systems.

The first of these is as follows. Let M_α be the set of monomials corresponding to the principal derivatives of the unknown z_α which occur in S. Let the monomials of M_α be multiplied by variables as in §36 so as to give the complete set M_α^*. Let the corresponding differentiations be performed on the appropriate equations of S and the resulting polynomials adjoined as equations to give S^*.

We endow an equation with the same multipliers as its leader.

The second modification is the following. If a polynomial g of S^* contains a derivative δy, where y is leader in an equation f, we have seen above that

δy is leader in δf. Since δy does not follow the leader of g, after a finite number of adjunctions, we obtain the differential system

$$S^* + \delta f + \cdots , \tag{45.2}$$

equivalent to S and having the property: if a principal derivative occurs in the system it is a principal indeterminate in the system, that is, it occurs as a leader in an equation. As seen at the beginning of this section, the process of reducing (45.2) to simple form involves merely reducing its polynomials with respect to each other. When (45.2) has been rendered *simple*, the system is called *standard*.

In one case, it is convenient to modify the requirements for a standard system by removing the restriction on the discriminant of an inequation. A system which is standard except possibly for the absence of this one property and in which corresponding to each unknown there is *at most one equation* with a derivative of that unknown for leader is of particular importance and is called *normal*.

46. **Passive systems.** Let a standard system S contain the equations f_1, f_2 with leaders y_1, y_2. From (45.2) we have

$$\delta_1 f_1 = f_1' \delta_1 y_1 + g_1 , \qquad \delta_2 f_2 = f_2' \delta_2 y_2 + g_2 , \tag{46.1}$$

where δ_1, δ_2 are any two differential symbols and the ordinals of g_1, g_2 are less than those of $\delta_1 y_1$ and $\delta_2 y_2$, respectively. If $\delta_1 = \partial/\partial x$, where x is non-multiplier for f_1, the fundamental property (§36) of complete sets shows that f_2 and δ_2 can be found to satisfy $\delta_1 y_1 = \delta_2 y_2$. With them so determined, we have the identity

$$f_2' \delta_1 f_1 - f_1' \delta_2 f_2 - f_2' g_1 + f_1' g_2 = 0, \tag{46.2}$$

so that S as a differential system implies the equation

$$f_2' g_1 - f_1' g_2 . \tag{46.3}$$

Let the totality of conditions (46.3) for all choices of f_1 and δ_1 be adjoined to S and the resulting system factored into standard systems. Any derivative which is leader in one of these systems but not in S is a parametric derivative in S. If the process of forming (46.2) be continued these derivatives form a sequence of sets such that no member of any set is a derivative of a member of any preceding set. By Theorem 22.1 the number of sets is finite. Hence after a finite number of steps the S under consideration, taken with its prolongations, implies as an *algebraic* system every equation (46.2). A standard system with this property will be called *passive*.

Since for a normal system there is just one monomial in each M^*, there are no non-multipliers and the conditions of passivity are vacuously satisfied. Hence we have

THEOREM 46.1. *Every normal system is passive.*

47. **Determined systems and the existence assumption.** Let $S(\xi)$ denote the system obtained from an algebraic system S by the substitution $x^i = \xi^i$,

where the ξ's are constants. By the change of independent variables (§24) $x^i = \bar{x}^i + \xi^i$, this substitution can be made $\bar{x}^i = 0$, so that we shall always assume for convenience $\xi^i = 0$.

If S is simple and $x^i \neq 0$ is not implied by it, $S(0)$ is a simple algebraic system (with coefficients in the ring of constants), which has solutions by Theorem 40.1, the unconditioned indeterminates being given arbitrary values.

An algebraic system S, whose coefficients are functions of the x's, is called *determined* when with it is associated (i) a set of constants which satisfy $S(0)$ and which are called a *numerical determination* for the system; (ii) a set of functions of x, called an *initial determination*, which are the values of the parametric indeterminates and which for $x = 0$ become the corresponding constants in the numerical determination. By a root of the determined system is meant a set of functions which satisfy S and which for $x = 0$ become the numerical determination, the parametric indeterminates being the functions prescribed by the initial determination.

We assume

E_0. *Every determined simple algebraic system has a unique root.*

This implies that if $y^1_\alpha(x)$ and $y^2_\alpha(x)$ are roots of S, if $y^1_\alpha(0) = y^2_\alpha(0)$, and if $y^1_\alpha(x) = y^\alpha(x)$ whenever y_α is parametric, then $y^1_\alpha(x) = y^2_\alpha(x)$ for all α's.

Consider now a standard system S. Select a root y of the system $S(0)$ regarded as an algebraic system in the unknowns and their derivatives, that is, select a numerical determination. Next choose an initial determination I to satisfy

$$\left(\frac{\partial z_\alpha}{\partial m}\right)_{x=0} = I_{\alpha m}(0), \tag{47.1}$$

where the left members are given by the numerical determination. When such an I has been associated with S, the resulting system is called *determined*. This definition becomes that given for an algebraic system if the only derivatives occurring in S have order zero.

By a *solution of a determined system* arising from S is meant a solution of S such that for every m in $\bar{M}_\alpha$

$$\left(\frac{\partial z_\alpha}{\partial m}\right)_{m'=0} = I_{\alpha m}, \tag{47.2}$$

where m' consists of the non-multipliers of m and $m' = 0$ means putting all the variables in m' equal to zero. We now make the following assumption, which contains E_0:

E. *Every determined normal system has a unique solution.*

If we subject the unknowns of a determined system S to the substitution

$$z_\alpha = z^*_\alpha + (\sigma^1)^{i_1} \cdots (\sigma^n)^{i_1} I_{\alpha m}, \tag{47.3}$$

where the i's are the indices of m, there results a system in the unknowns z^*_α, whose initial determination has the same form as that of S. Differentiation of (47.3) with respect to m and evaluation for $m' = 0$ gives, because of (47.2),

$$\left(\frac{\partial z^*_\alpha}{\partial m}\right)_{m'=0} = 0,$$

that is, the solution of S is transformed by (47.3) into a solution of S^* with zero initial determination, and vice versa. We state this as

THEOREM 47.1. *Every determined system can be written as a system whose initial determination consists of zeros by a change of unknowns.*

48. **Identities satisfied by equations of a passive system.** Let f be an equation of a passive standard system S and associate with it a new indeterminate F. We assign to F and to $F - f$ the cote of the leader of f, with an additional component as follows. The monomials of each set M^*_i are arranged according to rank. The $F - f$ whose leader corresponds to the monomial of highest rank in each M^*_i receives the component 1, the next highest 2, and so on. The addition to the cote of the independent variables is made zero.

Let T be the system containing all the polynomials $F - f$ as equations. T is then a simple algebraic system with the F's for conditioned indeterminates. If the unconditioned indeterminates are fixed by identifying the unknowns z with any symbols in $\mathfrak{R}$, the F's in the corresponding root of T give, of course, the values assumed by the equations f when the substitution in question is made on the z's.

Since a passive system implies all equations like (46.3), by Theorem 40.6 the result of reducing (46.3) by means of the system is zero. This process of reduction consists in replacing (46.3) by

$$a_0(f_2' g_1 - f_1' g_2) - hf, \tag{48.1}$$

where f is an equation of the prolongation of S, where $\bar{a}_0$ is implied by S, and where h is a polynomial; and in repeating the process on (48.1). Hence we find an identity of the form

$$A(f_2' g_1 - f_1' g_2) - HP = 0,$$

where P is a polynomial, every term of which contains an equation of S or one of its derivatives, and $\bar{A}$ is implied by S. This identity combined with (46.2) gives

$$Af_2'\delta_1 f_1 - Af_1'\delta_2 f_2 - HP = 0. \tag{48.2}$$

If in P, $\delta_1 f_1$, and $\delta_2 f_2$ we replace each equation f of S by the corresponding $F - f$, we get a relation satisfied by $F = f$, that is, by the values of the equations of S for all values of the unknowns. The aggregate of the terms involving

neither an F nor a derivative of an F, moreover, will be precisely the negative of the left member of (48.2). Hence use of (48.2) gives an identity of the form

$$Af_2'\delta_1F_1 - Af_1'\delta_2F_2 - K = 0, \tag{48.3}$$

where every term of K contains at least one F or a derivative of an F.

It is readily seen *that δ_2F_2 and any δF in K precede the corresponding δ_1F_1*.

49. **Decomposition of a standard system into normal systems.** In a determined standard system S let $S[0]$ denote the subset comprising all the inequations, all the polynomials like A in (48.3) as inequations and all the equations which have all the x's for multipliers.

Let $S[m']$ denote the system arising from the inequations of S and those equations which have the variables m' for non-multipliers by the following substitution. The non-multipliers are replaced by 0 and the derivatives

$$\left(\frac{\partial z}{\partial m}\right)_{m'=0} \tag{49.1}$$

are evaluated in a way now to be explained.

The symbol (49.1) is evaluated from $I_{\alpha p}$ if and only if $m = \lambda p$, where λ contains only multipliers of p and m' contains *all* the non-multipliers of p.

If (49.1) cannot be obtained as just indicated from the initial determination, let $m = \lambda\mu$, where λ is the maximum factor of m having no x in common with m'. Every x in μ then occurs in m'. Replace $(\partial z/\partial\mu)_{m'=0}$ by a new unknown ζ, so that (49.1) is replaced by $\partial\zeta/\partial\lambda$.

The new derivative is given the ordinal of the derivative from which it is obtained by evaluation. A leader in S will therefore give rise to a leader in $S[m']$. If two leaders $\partial z/\partial m_1$ and $\partial z/\partial m_2$ of equations in S are to give rise to leaders in the same system $S[m']$, m_1 and m_2 must have the same non-multipliers m'. If $\partial z/\partial m_1$ and $\partial z/\partial m_2$ are to give rise to derivatives of the same new unknown, $m_1 = \lambda_1\mu$ and $m_2 = \lambda_2\mu$. Since neither λ_1 nor λ_2 has an x in common with m' and m' comprises all the non-multipliers of m_1 and m_2, λ_1 and λ_2 contain only multipliers of m_1 and m_2. The exponent of each variable in λ_1 and λ_2 is therefore the maximum in the set M^* and $\lambda_1 = \lambda_2$. Hence, *$S[m']$ contains at most one equation whose leader is a derivative of a given unknown.*

The system $S[m']$ can readily be made normal (§45) in the new unknowns by adjoining to it the initials of all its polynomials as inequations, replacing the inequations of a given ordinal by a single inequation and reducing with respect to the equations. The normal system so obtained will be denoted by $S_{[m']}$.

We shall further use $S_{[i]}$ to denote the sum of all systems $S_{[m']}$ for which m' contains exactly i variables. This is consistent with the meaning already assigned $S_{[0]}$.

We have seen above the truth of

THEOREM 49.1. *Every determined standard system can be written as a sum of normal systems, all but one of which involve fewer independent variables.*

Consider an equation of $S_{[m']}$ with leader

$$\frac{\partial}{\partial \lambda}\left(\frac{\partial z_\alpha}{\partial \mu}\right)_{m'=0}. \tag{49.2}$$

The initial determination for it involves evaluating a number of derivatives of the form

$$\frac{\partial}{\partial \lambda'}\left(\frac{\partial z_\alpha}{\partial \mu}\right)_{m'=0}, \tag{49.3}$$

where λ' is a proper divisor of λ. Because of this, with reference to $S_{[m']}$, (49.3) has at least one non-multiplier from λ. To evaluate (49.3) at least one variable in addition to those of m' must be made zero and that variable is also non-multiplier with reference to S. Hence if (49.3) is principal in S, it can be evaluated from one of the unknowns of $S_{[i]}$, where $i > j$ and j is the number of variables in m'. If (49.3) is parametric, it can be evaluated from the initial determination for S. With the understanding that the normal systems are to be rendered determined in this manner, we have

THEOREM 49.2. *The normal systems $S_{[i]}$ of Theorem 49.1 become determined if an initial determination is associated with each of them by means of the original initial determination and the unknowns in systems $S_{[j]}$, where $i < j$.*

50. **The uniqueness theorem.** The uniqueness theorem is as follows:

THEOREM 50.1. *A determined standard system has at most one solution.*

The systems in $S_{[n]}$ are normal and determined. Hence they have a unique solution, which of course is part of the numerical determination. The systems in $S_{[n-1]}$ are made determined by taking their initial determinations as in §49 from that for S and from the unknowns in $S_{[n]}$. By E, $S_{[n-1]}$ has a unique solution.

Assuming $S_{[i]}$ has a unique solution, we see that $S_{[i-1]}$ is a determined normal system. Hence it has a unique solution. By induction, we conclude that $S_{[0]}$ has a unique solution. Since every solution of S satisfies $S_{[0]}$, the theorem has been proved. Whether or not the solution of $S_{[0]}$ satisfies S is another matter.

A simple system S can be factored into a finite number of standard systems. If the system is determined, all its solutions will satisfy one and only one of these standard systems, for the numerical determination of S can satisfy only one of the factors of S, which as algebraic systems have no root in common (Theorem 39.1). Hence we have

THEOREM 50.2. *A determined simple differential system has at most one solution.*

51. **The fundamental existence theorem.** The fundamental existence theorem is

THEOREM 51.1. *A determined passive standard system has a unique solution.*

By the analysis of the preceding section, there exists a solution of $S_{[0]}$ whose initial determination satisfies $S_{[i]}$, $i \neq 0$. We wish to show that the solution satisfies S by using the passivity conditions.

It is clear that the inequations of S are satisfied because they are included in $S_{[0]}$.

For the determination of the values assumed by the equations of S when the solution of $S_{[0]}$ is substituted in them, we have the equations like (48.3). Since $Af_2' \neq 0$, the equations (48.3) constitute a simple system. Let us now examine the initial determination of F_1. Since there is an equation (48.3) corresponding to each non-multiplier of f_1, the complementary set for F_1 consists of 1 with the same multipliers and non-multipliers as f_1. Hence the initial determination for F_1 is the value assumed by F_1 when the non-multipliers of f_1 are made zero. Since the initial determination of $S_{[0]}$ satisfies $S_{[i]}$, the initial determination of F_1 is zero and the system (48.3) is determined.

But since every term of K contains an F or a derivative of an F, it is obvious that (48.3) are satisfied by making all the F's zero. Since this set of values has the proper initial determination, the determined system is satisfied by making all the F's zero. By Theorem 50.2 there is at most one solution. Hence we conclude that all the F's are zero, that is, S is satisfied and the theorem is proved.

52. **Equivalence to Cauchy systems.** In a normal system S let f be an equation whose leader is $\partial z/\partial m$. Let x and y be variables in m such that $\partial/\partial x < \partial/\partial y$ and suppose their exponents in m are i, j, respectively. Consider the set of all derivatives $\partial z/\partial m'$, where the degree of m' equals that of m and the sum of the exponents of x and y in m' is $i + j$. Denote these derivatives by t_λ, λ being the exponent of x.

If we perform the change of variables

$$\bar{x} = x, \qquad \bar{y} = \alpha x + y, \tag{52.1}$$

then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \bar{x}} + \alpha \frac{\partial}{\partial \bar{y}}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial \bar{y}}, \tag{52.2}$$

and in the usual symbolic notation

$$\frac{\partial^{i+j}}{\partial x^i\, \partial y^j} = \left(\frac{\partial}{\partial \bar{x}} + \alpha \frac{\partial}{\partial \bar{y}}\right)^i \frac{\partial}{\partial \bar{y}^j}.$$

If $\bar{t}_0 = \partial^{i+j}/\partial \bar{y}^{i+j}$, direct computation gives

$$\frac{\partial}{\partial \bar{t}_0}\left(\frac{\partial^{i+j}}{\partial x^i\, \partial y^j}\right) = \alpha^i. \tag{52.3}$$

We suppose that each new $\bar{x}^i$ has the same cotes as the corresponding old x^i. It is clear that (52.2) preserves the order in x and y (that is, $i + j$) as well

as the order $(i_1 + i_2 + \cdots + i_n)$ in all the variables. Hence indirect differentiation and use of (52.3) give

$$\frac{\partial f^*}{\partial \bar{t}_0} = \left(\frac{\partial f}{\partial t_\lambda}\right)^* \frac{\partial t_\lambda}{\partial \bar{t}_0} = \alpha^\lambda \left(\frac{\partial f}{\partial t_\lambda}\right)^*,$$

where the asterisk denotes the result of making substitution (52.1).

If $\partial f/\partial t_0 \equiv 0$ and $\partial f^*/\partial \bar{t}_0 \equiv 0$ for all α's, we have

$$(52.4) \qquad \alpha^k \left(\frac{\partial f}{\partial t_k}\right)^* + \alpha^{k-1} \left(\frac{\partial f}{\partial t_{k-1}}\right)^* + \cdots + \alpha \left(\frac{\partial f}{\partial t_1}\right)^* = 0.$$

Dividing by α gives a polynomial in α which must be zero even for $\alpha = 0$ because it has an infinity of roots (Theorem 34.2 and Assumption A_8). Hence we may place $\alpha = 0$ after the division and get

$$\frac{\partial f}{\partial t_1} = 0.$$

Repetition gives

$$\frac{\partial f}{\partial t_\lambda} = 0,$$

so that no t_λ is leader. We are therefore forced to conclude that for some value of α

$$\frac{\partial f^*}{\partial \bar{t}_0} \not\equiv 0.$$

With such a choice of α the leader of f^* is $\geqq \bar{t}_0$.

If the leader is not $\bar{t}_0$ and the differentiation of the leader is not all with respect to the last independent variable, the operation can be repeated. It must ultimately end, however, because the ordinal of the derivatives involved is increasing and bounded. The leader of f is finally a derivative of the form $\partial^p z/\partial y^p$, where $\partial/\partial y$ follows $\partial/\partial x$ for every independent variable x except y.

The new system may not be normal, but it can be made normal by factorization. Alternate application of the two operations finally gives a factorization into Cauchy systems, which will now be defined.

A normal system is a *Cauchy system* if all the leaders of its equations involve only the last of the differential operators $\partial/\partial x$.

THEOREM 52.1. *Every normal system is the product of a finite number of Cauchy systems.*

Unfortunately, we do not have a method for finding the initial determination for the Cauchy system which will yield a given initial determination for the equivalent normal system. Such a method would, for example, render unnecessary a convergence proof to be given in Chapter X.

CHAPTER VIII

FUNCTION SYSTEMS AND DIFFERENTIAL SYSTEMS

The present chapter extends the theory of Chapters VI and VII to systems whose members are chosen from a class of functions more general than polynomials. Our main task is to make the necessary modifications in the definitions and processes so that the theory on the whole will remain true for the more general systems.

53. **Definition of the systems.** Let $\mathfrak{R}$ satisfy the assumptions of Chapter III and consist of symbols which are functions of r variables $y_1, \cdots, y_r$. A system S (§2) composed of functions from $\mathfrak{R}$ is called a *function system.*

If S has a root in the field of constants, it is called *determinable.* Such a root is, as previously, called a numerical determination, whose association with S gives a determined system.

By a change of variables the numerical determination can be made $y = 0$. We shall always assume that such a transformation has been made.

By the *ordinal of a function* is meant the highest index on a y appearing in it.

54. **The Weierstrass assumption.** The Weierstrass assumption is as follows.

W. *Every determined equation F is equivalent either to an equation of ordinal less than i or to a polynomial of $\mathfrak{R}_i[y_i]$, that is,*

$$F = fF_1, \tag{54.1}$$

where f is a polynomial

$$f = a_0 y_i^m + a_1 y_i^{m-1} + \cdots + a_m \tag{54.2}$$

whose coefficients are functions of $y_1, y_2, \cdots, y_{i-1}$; where the system $\bar{a}_0 + a_1 + \cdots + a_m + \bar{F}_1$ has the root $y_1 = \cdots = y_{i-1} = 0$, and where F_1 is inconsistent.

55. **The reduction process.** Because of W, any system can be replaced by a system with ordinals less than r plus a finite set of polynomials in y_r, which we shall denote as before by S_r.

The process of §39 is to be applied to the polynomials in S_r. There are, however, two important modifications which must be made in the reduction. In the first place, we notice that for $y_1 = \cdots = y_{i-1} = 0$ the f in (54.2) and its derivative f' reduce to

$$a_0 y_i^m, \quad m a_0 y_i^{m-1}, \tag{55.1}$$

respectively. Hence

THEOREM 55.1. *The discriminant of* (54.2) *vanishes for the numerical determination if and only if $m > 1$.*

Accordingly, we may ultimately have to exclude the origin from the scope of the variables, and in interpreting a sequence like

$$R = R_1 = \cdots = R_{p-1} = 0, \qquad R_p \neq 0$$

we understand that when some of the y's equal zero, R_p may be zero, but its vanishing for other values is not obviously implied by the system, and there exist values in the scope for which it is different from zero.

In the second place, reduction of a non-polynomial inequation by means of its common with an equation is impossible. A consequence of this is that after successive treatment of $S_r, S_{r-1}, \cdots, S_1$ a final factor may contain an equation (polynomial) and an inequation (non-polynomial) of exactly the same ordinal.

We assume

Z. *A determined function system S is equivalent to $S - T$, where T represents the non-polynomial inequations of S, provided the scope of the variables is properly restricted.*

When the non-polynomial inequations have been omitted, the final factors have all the features of simple systems and the terms *simple* and *canonical* will be applied to them.

In a simple function system S let the principal variables be $y_{n+1}, \cdots, y_r$, and the parametric be independent variables $x_1, \cdots, x_n$. By hypothesis, when the scope of the variables is restricted, there exist values $x_{10}, \cdots, x_{n0}$ which make S a simple algebraic system. Let these values be placed in a numerical determination $x_{10}, \cdots, x_{n0}, y_{n+1,0}, \cdots, y_{r0}$. We have then an initial determination for S because we have already disposed of the parametric variables. Assumption E_0 (as extended below) shows the existence of a unique root of S

$$y_\alpha(x_1, \cdots, x_n) \qquad (\alpha = n + 1, \cdots, r) \tag{55.2}$$

such that

$$y_\alpha(x_{10}, \cdots, x_{n0}) = y_{\alpha 0}. \tag{55.3}$$

From (55.1) we see that the jacobian J of the polynomials $f_{n+1}, \cdots, f_r$ for $x_1 = \cdots = x_n = 0$ becomes

$$m_{n+1} \cdots m_r a_{0n+1} \cdots a_{0r} y_{n+1}^{m_{n+1}-1} \cdots y_r^{m_r-1}.$$

Hence if $J \neq 0$, the polynomials are all linear. This gives the ordinary implicit function

Theorem 55.2. *If a determined function system of equations has a numerical determination for which its jacobian is not zero, it has a unique root.*

A further useful result is contained in

THEOREM 55.3. *If $\bar{f}_1, \bar{f}_2, \cdots, \bar{f}_r$ are consistent function systems, the system*

$$\bar{f}_1 + \bar{f}_2 + \cdots + \bar{f}_r = \overline{f_1 f_2 \cdots f_r}$$

is consistent.

The theorem is obvious and trivial for $r = 1$. Suppose $\bar{f}_1 + \bar{f}_2$ has a root. By W we may write

$$f_1 = \varphi_1 F_1,$$

where φ_1 is of degree m_1 in its leader y_1 and $\bar{\varphi}_1 \geqq \bar{f}_1$. If the parametric variables, i.e., those other than y_1, are assigned arbitrary values, φ_1 has at most m_1 zeros. If $\bar{\varphi}_1 + \bar{f}_2$ has a root, we may write in similar fashion

$$f_2 = \varphi_2 F_2.$$

It is obviously a matter of notation to assume that the parametric variables for φ_1 are parametric for φ_2. When those for φ_2 have been assigned arbitrary values, $\bar{\varphi}_2$ has by Theorem 35.1 more than m_1 roots. Consequently, not all the roots of $\bar{\varphi}_2$ are roots of φ_1, and $\bar{\varphi}_1 + \bar{\varphi}_2$ is consistent. So also is $\bar{f}_1 + \bar{f}_2$. If either $\bar{f}_1 + \bar{f}_2 = 1$ or $\bar{\varphi}_1 + \bar{f}_2 = 1$, the same result obviously holds, so that the theorem is true for $r = 2$.

That it is true for the general case then follows from an induction based on the identity

$$\bar{f}_1 + \cdots + \bar{f}_{r-1} + \bar{f}_r = \overline{f_1 \cdots f_{r-1}} + \bar{f}_r.$$

56. **Differential systems.** The passage from a function system to a differential system is exactly the same as from an algebraic system to an algebraic differential system. The definitions, processes, and theorems of Chapter VII are seen to hold when the word "system" is interpreted to mean "function system" instead of "algebraic system," and the word "normal" in E is given its extended meaning.

Reduction to passive form, of course, requires a process for recognizing whether a system is determinable or not. Even in the cases which we shall give as consistency examples an algorithm for finding a numerical determination does not exist. The applicability of the reduction process to given systems is, therefore, conditioned by our ability to judge whether or not numerical determinations exist and to find them in the affirmative case.

For convenience denote by B the whole set of assumptions which we have made. A ring $\mathfrak{R}$ which satisfies B will be called *normal* if for every choice of a finite set of symbols $y^1, \cdots, y^r$, none of which belongs to $\mathfrak{R}[x']$, the ring $\mathfrak{R}[x']$ is contained in a ring $\mathfrak{S}[x', y']$ which satisfies B. The ring $\mathfrak{S}[x', y']$ will be called an *extension* of $\mathfrak{R}[x']$. An important application of this definition follows.

If the initial determination for a differential system from a normal ring consists solely of the values z_0 of the unknowns for $x = 0$, it is called a *total* differential system. Let $\mathfrak{S}[x', z_0']$ be a ring satisfying B and containing $\mathfrak{R}[x']$. Let S

be a passive, standard system of total differential equations from $\mathfrak{R}[x']$. The totality of its solutions given by

$$z^\alpha(x, z_0), \tag{56.1}$$

where the z_0's are arbitrary constants, is called its *general solution.*

S can also be regarded as a passive, standard system in $\mathfrak{S}[x', z_0']$. Its initial determination is then the set of functions $f^\alpha(z_0^1, \cdots, z_0^r)$ to which the unknowns reduce for $x = 0$. If we choose $f^\alpha = z_0^\alpha$, we obtain a unique solution $Z^\alpha(x, z_0)$. If the z_0's are replaced in it by arbitrary values from the field of constants, $Z^\alpha(x, z_0)$ becomes a solution of the system S in $\mathfrak{R}[x']$ reducing to z_0^α for $x = 0$. Since this solution is unique, we must have

$$Z^\alpha(x, z_0) = z^\alpha(x, z_0),$$

that is, *the general solution belongs to $\mathfrak{S}$ because Z^α does.* We have therefore

THEOREM 56.1. *The general solution of a passive system of total differential equations in a normal ring belongs to every extension of the ring of dimension $n + r$, where n is the dimension of the normal ring and r is the number of unknowns.*

If $\mathfrak{R}$ consists of the functions analytic in x and $\mathfrak{S}$ of those analytic in x, z, the above theorem states that the general solution is analytic in the arbitrary constants as well as in the independent variables.

CHAPTER IX

PFAFFIAN SYSTEMS

If the members of S are chosen from a differential ring $\mathfrak{R}[u]$, rather than being restricted to the corresponding ring $\mathfrak{R}$, a pfaffian system results. Although the equivalence of such systems to the differential systems of Chapter VIII will ultimately be apparent, a direct study of pfaffian systems is interesting and profitable because of the symmetry which their theory possesses. The $\mathfrak{R}$ is that of Chapter IV. It is as a rule assumed to be a field.

57. **Integral varieties of a pfaffian system.** A set P of forms, labeled equations or inequations, from a differential ring $\mathfrak{R}[x']$ is called a *pfaffian system*, whose *degree* is the maximum degree of its forms. The pfaffian system treated here is really a generalized pfaffian system,[21] the ordinary case being the linear, some of whose properties are discussed in Chapter IV.

A substitution

$$(57.1) \qquad T: \quad x^i = f^i(t^1, \cdots, t^q),$$

where the t's are called parameters, is an *integral variety* of P if it converts every equation of P into the zero form and every inequation into a non-zero form in $\mathfrak{R}[t']$. The dimension of $\mathfrak{R}[t']$ is the *dimension* of T.

The totality of integral varieties is the content. Two systems are equivalent if they have the same content.

The prolonged system P' of P is obtained by adjoining to P the differential of every one of its equations. It is then clear that $(P')' = P'$, and since (24.9) holds when F is a form, rather than a symbol of $\mathfrak{R}'$, we have

THEOREM 57.1. *Every pfaffian system is equivalent to its prolonged system.*

58. **Fundamental formulas and identities.** Under the substitution (57.1) the form (8.1) becomes

$$F = a_{i_1 \cdots i_p} x^{i_1}_{\alpha_1} \cdots x^{i_p}_{\alpha_p} t'^{\alpha_1} \cdots t'^{\alpha_p} \qquad (i = 1, 2, \cdots, n;\ \alpha = 1, 2, \cdots, q),$$

where $x^i_\alpha = \delta_\alpha x^i$ in the ring $\mathfrak{R}[t']$. This result can be conveniently rewritten as

$$(58.1) \qquad F = F_{\alpha_1 \cdots \alpha_p} t'^{\alpha_1} \cdots t'^{\alpha_p},$$

where

$$(58.2) \qquad F_{\alpha_1 \cdots \alpha_p} = a_{i_1 \cdots i_p} x^{i_1}_{\alpha_1} \cdots x^{i_p}_{\alpha_p}.$$

The forms $F_{\alpha_1 \cdots \alpha_p}$ are associates (§12) of F.

[21] The idea of applying Cartan's methods to non-linear systems seems to be Goursat's ([10] and [9, 114]). Cerf considered the problem and gave a brief summary concerning inequalities on the dimension of an integral variety in [6]. An existence theorem with demonstration seems to have been published first by the author [19].

The differential F' can be written likewise as

$$F' = F'_{\alpha_0\alpha_1\cdots\alpha_p} t'^{\alpha_0} t'^{\alpha_1} \cdots t'^{\alpha_p}. \tag{58.3}$$

If (58.1) is differentiated and compared with (58.3), there results the fundamental identity

$$\frac{\partial F_{\alpha_1\cdots\alpha_p}}{\partial t^{\alpha_0}} - \cdots - (p+1)F'_{\alpha_0\alpha_1\cdots\alpha_p} = 0, \tag{58.4}$$

where the unwritten terms arise from the first by performing on it all signed transpositions of the form $(\alpha_0\alpha_i)$.

Replacing F by F' in this identity, or differentiating (58.3), gives the second fundamental identity

$$\frac{\partial F'_{\alpha_0\alpha_1\cdots\alpha_p}}{\partial t^{\alpha_{p+1}}} - \cdots = 0, \tag{58.5}$$

where the unwritten terms are obtained by applying the signed transpositions $(\alpha_{p+1}\alpha_\lambda)$ to the first term.

59. **The auxiliary differential system.** Corresponding to a pfaffian system P there exists an important differential system, which is satisfied by every one of its integral varieties and whose formation we proceed to describe.

In accordance with §55 let the equations of degree zero in P be replaced by a set of functions, each of which is a polynomial in its leader, and let the scope of the variables be restricted so that the discriminants are not zero. The resulting pfaffian system will be called *simple*.

Let F be a form of degree p. Write in some order those of its associates $F_{\lambda\alpha_2\cdots\alpha_p}$ whose indices α are less than λ and denote them by $f_{\lambda 1}, \cdots, f_{\lambda l}$. Place as equation in a system P_λ these associates for every equation F which is in P' and whose degree does not exceed λ. Corresponding to every inequation F in P replace P_λ by the product

$$(P_\lambda + \bar{f}_{\lambda 1})(P_\lambda + f_{\lambda 1} + \bar{f}_{\lambda 2}) \cdots (P_\lambda + f_{\lambda 1} + \cdots + f_{\lambda, l-1} + \bar{f}_{\lambda l}). \tag{59.1}$$

It is thus seen that every integral variety of a pfaffian system satisfies a differential system of the form

$$S = P_0^* + P_1^* + \cdots + P_n^*, \tag{59.2}$$

which we may also write

$$S = \Pi_0\Pi_1 \cdots \Pi_l \tag{59.3}$$

by choosing all possible combinations of factors from the terms of (59.2). S will be called the *auxiliary system* for P.

The system to be satisfied by an integral variety of dimension γ is obtained by replacing the original pfaffian system P by

$$(P + \bar{\varphi}_{\gamma 1})(P + \varphi_{\gamma 1} + \bar{\varphi}_{\gamma 2}) \cdots (P + \varphi_{\gamma 1} + \cdots + \varphi_{\gamma, l-1} + \bar{\varphi}_{\gamma l}), \tag{59.4}$$

where $\varphi_{\gamma 1}, \cdots, \varphi_{\gamma l}$ denote the unit monomials of degree γ formed from $x'^1, \cdots, x'^n$, obtaining (59.2) as above, and finally setting $x^i_\lambda = 0$ for $\lambda > \gamma$. The corresponding S will be called the *auxiliary system for dimension* γ.

The reduction process of Chapter VIII can be applied to each factor in (59.3) and the system finally expressed as the product of passive standard systems. We have, therefore, a method for determining all the integral varieties of a pfaffian system. Positive existence theorems about integral varieties of a definite type will be given later (§61).

Instead of considering an auxiliary system which is purely differential, it is convenient at times to consider the auxiliary system

$$P_0 + P_1 + \cdots + P_n + \bar{F}_1 + \bar{F}_2 + \cdots + \bar{F}_k,$$

where the P's are as defined above and the F's are the inequations of the pfaffian systems. In this way, the factorizations (59.1) and (59.4), which may be awkward, can be avoided.

60. **Numerical determinations.** We consider in the next few sections a pfaffian system P containing only equations. A numerical determination for P_λ (considered as a function system) will be called *of dimension* λ and denoted by N_λ, if it makes the rank of $\| x^i_\alpha \|$ $(\alpha = 1, \cdots, \lambda)$ equal λ. If the values of x^i_λ are omitted from N_λ, a unique numerical determination of dimension $\lambda - 1$ results. We thus obtain a sequence

$$N_0 \subset N_1 \subset \cdots \subset N_{\lambda-1} \subset N_\lambda, \tag{60.1}$$

of which each N is said to be *imbedded* in those following it.

Let the part of N which is of dimension $\lambda - 1$ and involves differentiation with respect to $t^1, \cdots, t^\lambda$ be substituted in P_λ, which becomes thereby a linear, homogeneous system $L_\lambda(N)$ in the indeterminates x^i_λ, the rank of whose left members will be called $\rho_\lambda(N)$.

The successive *characters of a numerical determination* N_λ are defined by

$$\sigma_\alpha(N) = \rho_\alpha(N) - \rho_{\alpha-1}(N) \qquad (\alpha = 0, 1, \cdots, \lambda), \tag{60.2}$$

ρ_{-1} being interpreted as zero.

It is clear that if $N_\lambda \subset N$, then

$$\rho_\alpha(N_\lambda) = \rho_\alpha(N), \qquad \sigma_\alpha(N_\lambda) = \sigma_\alpha(N) \qquad (\alpha = 0, 1, \cdots, \lambda).$$

The linear homogeneous system $L_{\alpha+1}(N_\alpha)$ obviously has the α independent solutions

$$x^i_{\alpha+1} = (x^i_\beta)_0 \qquad (\beta < \alpha + 1), \tag{60.3}$$

where the symbol $(\ )_0$ denotes that the values belong to N_α. Hence we have

$$n - \rho_{\alpha+1}(N_\alpha) \geqq \alpha. \tag{60.4}$$

On the other hand,

$$\rho_{\alpha+1}(N') \geqq \rho_\alpha(N) \qquad (N \subset N'). \tag{60.5}$$

Since the right member of (60.4) increases with α and the left is non-increasing, sequence (60.1) must terminate with a $\gamma(N)$ satisfying

$$(60.6) \qquad n - \rho_{\gamma+1}(N) = \gamma(N).$$

From (60.6) and the definition of the σ's, we get

$$(60.7) \qquad n - \gamma(N) = \sigma_0(N) + \sigma_1(N) + \cdots + \sigma_{\gamma+1}(N).$$

61. **Non-singular integral varieties.** For simplicity, we again suppose that (59.3) has only one factor. Reduce it to a product of canonical factors, employing (2.6) exactly as it is written,

$$(61.1) \qquad S = A_0 A_1 \cdots A_l,$$

so that the extreme left factor A_0 has its special significance.

For the reduction, we assume that the symbols $\partial/\partial t^\alpha$ have been ordered so that $\partial/\partial t^\alpha > \partial/\partial t^\beta$ if and only if $\alpha > \beta$.

Since the equations are linear in their leaders, A_0 arises from Π_0 by the adjunction of inequations and the omission of equations. Let $A_{0\lambda}$ be members of A_0 which have derivatives x_λ^i for the given λ as leaders. Let the number of equations in $A_{0\lambda}$ be r_λ. Let N_λ be a numerical determination for $A_{0\lambda}$. As in §60 we infer the existence of an integer g such that

$$n - r_{g+1} = g,$$

and no N_{g+1} exists. g is called the *genus*.

The numbers r_α are employed ($r_{-1} = 0$) to define the *characters of the pfaffian system*

$$(61.2) \qquad s_\alpha = r_\alpha - r_{\alpha-1} \qquad (\alpha = 0, 1, \cdots, g + 1).$$

We have also as companion formula to (60.7)

$$(61.3) \qquad n - g = s_0 + s_1 + \cdots + s_{g+1}.$$

A numerical determination N_λ is called *non-singular* if

$$(61.4) \qquad \sigma_\alpha(N_\lambda) = s_\alpha \qquad (\alpha = 0, 1, \cdots, \lambda + 1).$$

An integral variety likewise is *non-singular* if it has at least one non-singular numerical determination.

If we compute $\rho_\alpha(N_g)$ for $A_{0\alpha}$, we obviously get r_α, so that (61.4) hold for $A_{0\lambda}$. Accordingly, *every integral variety of A_0 is non-singular for P.*

On the other hand, the resultant of at least one pair of the original equations in $P_0 + P_1 + \cdots$ is put equal to zero in A_i ($i > 0$). This implies the vanishing of at least one determinant which is different from zero in A_0. Hence, some $\rho_\alpha(N_g) < r_\alpha$ and *all g-dimensional varieties of A_i for $i > 0$ are singular.* For this reason, A_0 will be called *the non-singular auxiliary system* for P.

The determination of the integral varieties for P, therefore, is accomplished by reducing the factors of S to passive, standard form. In the case of A_0,

this further reduction is never necessary, as we shall now show by proving that A_0 is passive.

Denote by N_g a numerical determination for A_0. Let X_0 denote the corresponding N_0. Let the principal unknowns in A_{01} be assigned as initial determinations the corresponding values in X_0. Let the parametric unknowns be assigned arbitrary initial determinations such that x^i and x^i_α reduce for $t^1 = 0$ to the corresponding values in X_0. The system A_{01} is normal and has therefore a unique solution $X_1(t^1)$.

In general, suppose $X_{\lambda-1}(t^1, \cdots, t^{\lambda-1})$ a solution of $A_{0,\lambda-1}$ whose numerical determination is $N_{\lambda-1}$. Let the principal unknowns in $A_{0\lambda}$ be assigned as initial determinations the corresponding values in $X_{\lambda-1}$ and the parametric, arbitrary functions of $t^1, \cdots, t^\lambda$ which reduce for $t^\lambda = 0$ to the corresponding values in $X_{\lambda-1}$. The determined normal system $A_{0\lambda}$ has a unique solution which will be denoted by $X_\lambda(t^1, \cdots, t^\lambda)$.

In this way is constructed a sequence $X_\lambda(t^1, \cdots, t^\lambda)$ $(\lambda = 0, 1, 2, \cdots, g)$ such that

$$X_\lambda(t^1, \cdots, t^{\lambda-1}, 0) = X_{\lambda-1}(t^1, \cdots, t^{\lambda-1}) \tag{61.5}$$

and $X_\lambda(t^1, \cdots, t^\lambda)$ satisfies $A_{0\lambda}$. We shall show that $X_\lambda(t^1, \cdots, t^\lambda)$ satisfies $A_{0\mu}$ $(\mu = 0, 1, \cdots, \lambda - 1)$.

Let G be the result of substituting X_λ in an equation f of $A_{0\mu}$. f is of one of two types: it arises from either an $F_{\alpha_1\alpha_2\cdots\alpha_p}$ or an $F'_{\alpha_0\alpha_1\cdots\alpha_p}$, with every index $\leqq \mu$, in P_μ. Let us use (58.4) with $\mu < \alpha_0 \leqq \lambda$ in the first case and (58.5) with $\mu < \alpha_{p+1} \leqq \lambda$ in the second. In both cases, every term except the first contains an F with an index greater than μ. Since the F's are implied as equations by $A_{00} + \cdots + A_{0\lambda}$, the reduction of them by that system gives zero, that is, the identity (58.4) or (58.5) gives

$$B\frac{\partial G}{\partial t^\rho} + K = 0 \qquad (\rho = \mu + 1, \cdots, \lambda),$$

where A_0 implies $\bar{B}$ and K is a polynomial, every term of which involves either a symbol like G or the derivative of a symbol like G with respect to t^ρ $(\rho \leqq \mu)$. We have also

$$G(t^1, \cdots, t^\mu, 0, \cdots, 0) = 0,$$

since $X_\lambda(t^1, \cdots, t^\mu, 0, \cdots, 0)$ satisfies $A_{0\mu}$, by hypothesis. The quantities like G therefore satisfy a standard system, which is also satisfied by making every $G = 0$. The solution being unique by Theorem 50.1, we conclude as in §51 that every $G = 0$. This completes the induction and establishes the desired point because assumption Z takes care of the inequations.

We have accordingly extended Cartan's theorem [5] to non-linear systems.

THEOREM 61.1. *There is at least one integral variety of dimension* λ *containing a given non-singular integral variety of dimension* $\lambda - 1$ *for any* λ *not exceeding*

the genus. In particular, there is at least one non-singular integral variety of dimension equal to the genus and the totality of such varieties is the content of A_0.

X_λ for $\lambda < g$ will not satisfy A_0, but will be found among the solutions of some other factor A. These varieties, however, can readily be deduced from the integral varieties of maximum dimension.

System A_{00} can be solved for s_0 of the x's, which proper choice of notation will make $x^1, \cdots, x^{s_0}$. By the implicit function theorem, the equations $df^1, \cdots, df^{s_0}$, which occur in A_{01}, have rank s_0 with respect to $x_1^1, \cdots, x_1^{s_0}$. The non-vanishing determinant of order s_0 is contained by the result at the end of §11 in a non-vanishing determinant of order $s_0 + s_1$ in A_{01}. Hence A_{01} can be solved for $x_1^1, \cdots, x_1^{s_0}, x_1^{s_0+1}, \cdots, x_1^{s_0+s_1}$. Continuing, we see that A_0 can be put in the form of a regular system [20]. It is passive because of the solution which it has been shown to possess.

A sequence X_λ can be constructed for each of the other factors A_i $(i > 0)$, but the above method of proving that X_λ satisfies $A_{i\mu}$ $(\mu < \lambda)$ fails because $A_{i\mu}$ contains equations not implied by $P_0 + \cdots + P_\mu$. The factors are in general not passive, as can be shown by examples.

Another existence theorem is

THEOREM 61.2. *If* $X_\lambda(t^1, \cdots, t^{\lambda-1}, t^\lambda)$ *is a* λ*-dimensional solution of* P_λ *for* $\lambda = 0, 1, \cdots, \gamma$ *and if*

$$X_\lambda(t^1, \cdots, t^{\lambda-1}, 0) = X_{\lambda-1}(t^1, \cdots, t^{\lambda-1}),$$

then $X_\lambda(t^1, \cdots, t^{\lambda-1}, t^\lambda)$ *is a solution of* P_μ $(\mu \leqq \lambda)$ *and defines a* λ*-dimensional integral variety of the pfaffian system* P.

The proof follows much the same lines as that of the preceding theorem. The solution

$$X_\lambda(t^1, \cdots, t^{\lambda-1}, t^\lambda) \tag{61.6}$$

clearly satisfies P_λ. To employ induction, suppose that it satisfies $P_\lambda, P_{\lambda-1}, \cdots, P_{\mu+1}$.

Let G be the result of substituting (61.6) in one of the equations f of P_μ. f is one of two types: it is either an $F_{\alpha_1\alpha_2\cdots\alpha_p}$ or an $F'_{\alpha_0\alpha_1\cdots\alpha_p}$, with every index $\leqq \mu$. In the first case we employ (58.4) with $\mu < \alpha_0 \leqq \lambda$ and in the second, (58.5) with $\mu < \alpha_{p+1} \leqq \lambda$. In both cases, every term except the first has an index greater than μ and therefore vanishes by assumption. Hence it is clear that

$$\frac{\partial G}{\partial t^\rho} = 0 \qquad (\rho = \mu + 1, \cdots, \lambda).$$

Moreover,

$$G(t^1, \cdots, t^\mu, 0, \cdots, 0) = 0,$$

since $X_\lambda(t^1, \cdots, t^\mu, 0, \cdots, 0) = X_\mu(t^1, \cdots, t^\mu)$ satisfies $P_\mu(0)$, by hypothesis. Hence G satisfies a determined standard system, which is also satisfied by

$G = 0$. The solution being unique by Theorem 50.1 we conclude as in §51 that $G = 0$.

The induction is therefore finished and the theorem completely proved.

Theorem 61.2 has much in common with that published by C. Burstin[22] for the linear case. Its usefulness is limited because it neither asserts positively the existence of any integral varieties nor is it capable of giving all of them.

62. **Function systems as pfaffian systems.** A function system in the variables $x_1, \cdots, x_n$ can be regarded as a pfaffian system P of degree zero. We assume that P is simple and that the equations $f_1, f_2, \cdots, f_{r_0}$ have leaders $x^1, x^2, \cdots, x^{r_0}$. The set P' is found by adjoining

$$df_1, df_2, \cdots, df_{r_0},$$

and P_λ has for matrix the functional matrix of the f's. Now the functional determinant is

$$\frac{\partial f_1}{\partial x^1}\frac{\partial f_2}{\partial x^2}\cdots\frac{\partial f_{r_0}}{\partial x^{r_0}}$$

because $\partial f_i/\partial x^j = 0$ if $i > j$. It is different from zero for any numerical determination giving non-vanishing discriminants. Hence we have

$$r_0 = r_1 = \cdots = r_n, \qquad s_1 = s_2 = \cdots = s_n = 0. \tag{62.1}$$

The condition for a non-singular numerical determination is, therefore, that the jacobian be different from zero for the numerical determination. On identifying $x^{r_0+1}, \cdots, x^n$ with the parameters we find from Theorem 55.2

THEOREM 62.1. *Every simple pfaffian system of degree zero which has been rendered determined by the adjunction of a non-singular numerical determination has a unique solution expressing certain of the independent variables in terms of the others.*

63. **Inequalities satisfied by the genus and characters.** When the pfaffian system is linear and $s_0 = 0$, we clearly have $s_1 \geqq s_\alpha$ because s_1 rows are added to the matrix in passing from $P_{\alpha-1}$ to P_α. Combined with (61.3) this gives Cartan's lower limit for the genus

$$g \geqq \frac{n - s_1}{s_1 + 1}. \tag{63.1}$$

In the linear case it can also be shown *that the characters form a non-increasing sequence*

$$s_{\lambda+1} \leqq s_\lambda . \tag{63.2}$$

[22] C. Burstin, *Ein Beitrag zur Theorie der Systeme Pfaff'scher Aggregate*, Recueil Mathématique de la Société Mathématique de Moscou, vol. 37 (1930), pp. 13–21. Cf. also J. M. Thomas, *The condition for a pfaffian system in involution*, Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 316–320.

To do this, we remark that P_λ is

$$a_i^\alpha a_\lambda^i = 0, \qquad a_{ij}^\alpha x_\mu^i x_\lambda^j = 0 \qquad (\mu < \lambda), \tag{63.3}$$

and that the equations of $P_{\lambda+1}$ can be obtained (i) by replacing λ by $\lambda + 1$, a process which adds no new elements to the matrix, (ii) by replacing μ by $\lambda + 1$, a process which adds

$$a_{ij}^\alpha x_\lambda^i x_{\lambda+1}^j = 0.$$

Suppose that $s_{\lambda+1} > s_\lambda$, and consider the matrices

$$M_\lambda = \begin{Vmatrix} a_{ij}^\alpha & x_\mu^i \\ a_{ij}^\alpha & x_{\lambda-1}^i \end{Vmatrix}, \qquad M_{\lambda+1} = \begin{Vmatrix} a_{ij}^\alpha & x_\mu^i \\ a_{ij}^\alpha & x_{\lambda-1}^i \\ a_{ij}^\alpha & x_\lambda^i \end{Vmatrix}, \qquad (\mu < \lambda - 1), \tag{63.4}$$

in which a combination α, μ (or α, $\lambda - 1$ or α, λ) determines the row and j the column. By hypothesis in $M_{\lambda+1}$ there are $s_{\lambda+1}$ rows which have index λ on the x and which are independent of the other rows. In particular, they must be independent of the rows with a μ on the x. Hence in the matrix

$$M'_\lambda = \begin{Vmatrix} a_{ij}^\alpha & x_\mu^i \\ a_{ij}^\alpha & x_\lambda^i \end{Vmatrix}$$

there are $s_{\lambda+1}$ rows which have index λ and which are independent of those with μ. Had we chosen the values x_λ^i instead of $x_{\lambda-1}^i$ in solving $P_{\lambda-1}$, we should have found the λth character to be $s_{\lambda+1} > s_\lambda$. This contradiction establishes (63.2).

Neither (63.1) nor (63.2) holds for a generalized pfaffian system. Similar inequalities, however, can be deduced, but they are complex in nature because of the number of elements involved. Cerf [6] has given some results bearing on this question.

64. **Calculation of the characters.** Let $\omega^\alpha = a_i^\alpha u^i$ be a set of r linear forms and F a form of degree p. Let

$$\Omega = \omega^1 \cdots \omega^r, \qquad G = \Omega F.$$

From (11.23) we have

$$Du^{i_k} = B_\alpha^{i_k} \omega^\alpha + C_\lambda^{i_k} u^\lambda, \tag{64.1}$$

where the B's and C's are polynomials in the a's and the range of λ is $1, 2, \cdots, n$ with the values $i_1, \cdots, i_r$ omitted. Let q be the smaller of p and r. On the right of the equation

$$D^q G = \Omega D^q F$$

every u^{i_k} can be coupled with a D and the substitution (64.1) can be applied to $D^q F$. Because Ω is present, the ω's can be neglected in making the substitutions, which accordingly give

$$D^q G = \Omega F^*, \tag{64.2}$$

where F^* involves only u^λ and has for coefficients polynomials in the a's and the coefficients of F.

Since

$$\Omega = Du^{i_1} \cdots u^{i_r} + \cdots, \tag{64.3}$$

where the unwritten terms involve fewer than r of the marks $u^{i_1}, \cdots, u^{i_r}$, differentiation of (64.2) gives

$$D^q \frac{\partial G}{\partial m} = DF^* \qquad (m = u^{i_1} \cdots u^{i_r}). \tag{64.4}$$

The coefficients on the left are polynomials divisible by D^q. The same must be true of those on the right, so that the coefficients of F^* are divisible by D^{q-1}. When we write $F^* = D^{q-1}\Phi$, relations (64.2) and (64.4) become

$$DG = \Omega\Phi, \tag{64.5}$$

$$\Phi = \frac{\partial G}{\partial m}. \tag{64.6}$$

Use of Theorem 13.1 and (64.5) gives

THEOREM 64.1. *If $D \neq 0$, the system*

$$\omega^\alpha = 0, \qquad F = 0$$

is equivalent to

$$\omega^\alpha = 0, \qquad \Phi = 0.$$

The foregoing result can be applied in computing the characters of a pfaffian system of degree p.

Let Φ^1 comprise the linear members of P' and let φ^1_α be the associates of Φ^1 with index α. The rank of φ^1_α is then $r_1 = s_0 + s_1$ for any α.

In the above discussion, let $\omega^1, \cdots, \omega^r$ be interpreted as r_1 independent members of Φ^1 and let F be in turn each non-linear form in P'. Let the corresponding set of Φ's furnished by application of Theorem 64.1 be Φ^2 and let φ^2_α contain their associates which have one index equal to α and no index exceeding α. Since the rank of $\varphi^1_2 + \varphi^2_2$ is $r_2 = s_0 + s_1 + s_2$ and that of φ^1_2 is $s_0 + s_1$, the rank of φ^2_2 is s_2. Let s_2 independent members of φ^2_2 be $A^\alpha_j x^j_2$. The forms $A^\alpha_j u^j$ have rank s_2. Let $\omega^1, \cdots, \omega^r$ be augmented by them and Theorem 64.1 applied to the set Φ^2 as a source of F's. Denote by Φ^3 the set of Φ's resulting. Let φ^3_α contain the associates of Φ^3 which have one index equal to α and none exceeding α. The rank of φ^3_3 is then s_3.

The general process should now be clear. It gives the characters by the use of ring operations alone.

65. **Systems comprising a single linear equation.** Let $r_0 = 0$, $r_1 = 1$, with ω representing the single linear equation as in §27. Since $s_1 = 1$, from (63.2) it follows that the sequence of characters consists of a number of 1's followed, if at all, by a number of zeros.

Let ρ be defined by

$$(65.1)\qquad \omega'^{\rho}\omega \neq 0, \qquad \omega'^{\rho+1}\omega = 0$$

in the sense that some coefficient in the inequation is different from zero for some values in the scope of the x's, whereas every coefficient in the equation is zero for all values of the x's.

Let a non-zero coefficient of $\omega'^{\rho}\omega$ be adjoined to A_0 (§61) as inequation. Since A_0 contains no equation with an x for leader, the resulting system is determinable by Theorem 55.3, and by Assumption Z it has a solution.

The reduction (§27) of the system ω to the canonical form

$$(65.2)\qquad z' + p^{\alpha}x'^{\alpha} \qquad (\alpha = 1, 2, \cdots, \rho)$$

is on the assumption (65.1). If a substitution is performed on the variables of the pfaffian system, the canonical form is invariant and the new canonical variables can be obtained by performing the substitution on the old provided (65.1) continue to hold. It has been seen that the adjunction of inequation (65.1) to system A_0 (§61) gives a consistent system. Consequently, *reduction to* (65.2) *is valid on some non-singular integral variety of dimension* g.

If $z = -\frac{1}{2}x^{\alpha}x^{\alpha}$, $p^{\alpha} = x^{\alpha}$ and $x^{\alpha}_{\beta} = p^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}$ $(\alpha, \beta = 1, 2, \cdots, \rho)$, the systems P_{λ} $(\lambda \leqq \rho)$ are seen to be satisfied. The dimension of this variety is obviously ρ. Each system P_{λ} contains the equation

$$x^{\lambda-1}_{\lambda} - x^{\lambda}_{\lambda-1} = 0,$$

which is independent of the others. Hence at least $\rho + 1$ characters are equal to 1. On the other hand, if we multiply the equation (65.2) by $u^1 \cdots u^{\rho}$ (for convenience, we shall abbreviate the differentials x' as u and p' as v) and the equation $v^{\beta}u^{\beta}$ by $\partial(u^1 \cdots u^{\rho})/\partial u^{\alpha}$, we find

$$(65.3)\qquad z'u^1 \cdots u^{\rho} = 0, \qquad v^{\alpha}u^1 \cdots u^{\rho} = 0,$$

so that the functional matrix has rank ρ and at most $\rho + 1$ characters are unity.

THEOREM 65.1. *If the class of a single linear pfaffian equation is* $2\rho + 1$, *its first* $\rho + 1$ *characters are unity and the rest zero. Its genus is* $n - \rho - 1$.

The foregoing analysis also serves to prove

THEOREM 65.2. *If the class of the linear equation* ω *is* $2\rho + 1$, *there must be at least one coefficient of* $\omega'^{\rho}\omega$ *which does not vanish on a given non-singular integral variety of maximum dimension.*

To prove, simply remark that if all the coefficients of $\omega'^{\rho}\omega$ are zero, there exists an algebraic reduction to

$$\omega' = v^{\alpha}u^{\alpha} \qquad (\alpha = 1, 2, \cdots, \rho' < \rho),$$

where the u's and v's are differential forms but not necessarily differentials. The computation of the numbers (§61) $\sigma(N)$ can be effected by the process of §64, which clearly gives fewer than $\rho + 1$ of them equal to unity.

Let the rank of the u's on an integral variety be λ and suppose the notation adjusted so that

$$u^1 \cdots u^\lambda \neq 0, \qquad u^1 \cdots u^\lambda u^\alpha = 0.$$

Multiplying the equation

$$z' + p^\alpha u^\alpha \tag{65.4}$$

by $u^1 \cdots u^\lambda$ gives $z'u^1 \cdots u^\lambda = 0$. Hence ($\mathrm{I}_2$) the symbols z, $x^{\lambda+1}, \cdots, x^\rho$ are functions of $x^1, \cdots, x^\lambda$. Taking the latter set as parameters $t^1, \cdots, t^\lambda$ and substituting in (65.4) gives

$$(z_\mu + p^\mu + p^\theta x^\theta_\mu)u^\mu \qquad (\mu = 1, 2, \cdots, \lambda;\ \theta = \lambda + 1, \cdots, \rho).$$

Multiplying by $\partial(u^1 \cdots u^\lambda)/\partial u^\mu$ gives

$$p^\mu = -z_\mu - p^\theta x^\theta_\mu. \tag{65.5}$$

The integral variety is thus completely determined.

Conversely, if z, $x^{\lambda+1}, \cdots, x^\rho$ are assumed as arbitrary functions of $x^1, \cdots, x^\lambda$, formulas (65.5) give an integral variety on which the parameters are $x^1, \cdots, x^\lambda$, $p^{\lambda+1}, \cdots, p^\rho$. If these parameters are taken as independent, an integral variety of dimension ρ results. By performing on it the inverse of the transformation leading to (65.2), the equations defining it in terms of the original dependent variables are obtained.

The variety so obtained may be singular or non-singular. For example, $z = -x$ is singular for $dz + pdx$.

THEOREM 65.3. *The non-singular integral varieties of maximum dimension for a single linear equation can be obtained from the canonical form by differentiation and ring operations alone.*

66. **Passive linear systems.** In this section we shall prove

THEOREM 66.1. *In a normal ring assumption* E *implies* I_3.

Let P be linear and contain $n - 1$ u's independent with respect to $u^1, \cdots, u^{n-1}$, so that it may be written

$$\omega^\alpha = u^\alpha + B^\alpha u^n \qquad (\alpha = 1, 2, \cdots, n-1). \tag{66.1}$$

The products $\omega^1 \cdots \omega^{n-1}\, \omega'^\alpha$ are zero because they are of degree $n + 1$ in n differentials. Hence (§64) s_2 and all subsequent characters are zero. The variable x^n is parametric. Let it be t^1, and assign a non-singular set of initial values x_0 to the other x's. The pfaffian system then has by Theorem 61.1 a non-singular integral variety defined by the equations

$$-x^\alpha + f^\alpha(x^1_0, \cdots, x^{n-1}_0, x^n) = 0 \qquad (\alpha = 1, 2, \cdots, n-1). \tag{66.2}$$

Regard this system as having the principal variables $x^1_0, \cdots, x^{n-1}_0$ and denote its equations by g^α. We have

$$\frac{\partial g^\alpha}{\partial x^\beta_0} = \frac{\partial f^\alpha(x^1_0, \cdots, x^{n-1}_0, x^n)}{\partial x^\beta_0},$$

whence on evaluating for $x^n = x_0^n$ and using (24.12) we get

$$\frac{\partial g^\alpha}{\partial x_0^\beta} = \frac{\partial f^\alpha(x_0^1, \cdots, x_0^{n-1}, x_0^n)}{\partial x_0^\beta} = \frac{\partial x_0^\alpha}{\partial x_0^\beta} = \delta_\beta^\alpha,$$

since f^α becomes x_0^α for $x^n = x_0^n$. Hence the jacobian of (66.2) is 1 for the initial determination. By Theorems 55.2 and 56.1 system (66.2) can be solved in the form

$$\varphi^\alpha(x^1, \cdots, x^n) - x_0^\alpha = 0 \qquad (\alpha = 1, 2, \cdots, n-1).$$

We have the identity

$$dx_0^\alpha = \frac{\partial x_0^\alpha}{\partial x^\beta}(u^\beta + B^\beta u^n) + \left(\frac{\partial x_0^\alpha}{\partial x^n} - B^\beta \frac{\partial x_0^\alpha}{\partial x^\beta}\right)u^n. \tag{66.3}$$

If x_0^α are regarded as constants in (66.2) and the f^α are substituted for x^α in (66.3), we get

$$\frac{\partial x_0^\alpha}{\partial x^n} = B^\beta \frac{\partial x_0^\alpha}{\partial x^\beta}. \tag{66.4}$$

These relations, proved only for the values (66.2), must be identities in x because the values to which f^α reduce for $x^n = x_0^n$ can be chosen arbitrarily. Hence the last expression disappears from the right of (66.3), which accordingly furnishes $n - 1$ differentials of the pfaffian system. They are independent because the coefficient of $u^1 \cdots u^{n-1}$ in the product $x_0'^1 \cdots x_0'^{n-1}$ is unity for $x^n = x_0^n$. The theorem is therefore true as stated.

CHAPTER X

CONSISTENCY EXAMPLES

We shall now prove the consistency of the assumptions by giving specific instances in which they are realized. Assumptions A_1 to A_5, D and I are shown to be valid if $\mathfrak{R}$ is interpreted as a set of functions having continuous second partial derivatives. The principal assumption about systems of differential equations is, of course, I_3 and the existence theorem upon which the corresponding consistency example is based goes back to Cauchy, although important simplifications have been made in the proof by Picard [14, 368]. The remaining assumptions are shown to hold if the ring is specialized to a set of holomorphic functions. The main postulate in this case is E and the existence theorem is Riquier's for orthonomic systems. The demonstration given here embodies not only the material simplifications previously introduced by Janet [12] and by the author [21], but also a simpler convergence proof, thanks to the employment of normal systems. It is to be noted, however, that all the simplifications consist in rearrangements and omissions without the addition of any essentially new principle. It is interesting to remark that except for one slight feature the convergence demonstration is now practically that given by Goursat [8, II, 374] for Cauchy systems of the first order.[23]

67. **The differentiation process.** Let $x^1, \cdots, x^n$ be n independent real variables in the ordinary sense. Let the symbol f represent a function (in the usual sense) of these variables having partial derivatives of every order not exceeding l continuous in some region

$$(67.1) \qquad | x^i - x_0^i | \leqq \rho,$$

where the ρ denotes a positive constant. The function f is said to be *of class* l.

Note that the functional relation is transitive and is therefore consistent with the definition in §1.

Let $f_1, \cdots, f_k$ denote a finite set of functions all of which are of class l in the *same* region (67.1). Any compound symbol formed by a finite number of additions and multiplications applied to the symbols $f_1, \cdots, f_k$ is readily proved to represent a function of class l in (67.1). The totality of such symbols forms a ring. The corresponding quotient field [23, I, 46] consists of symbols representing functions each of which admits a region like (67.1), in which it is of class l. For if a polynomial in the f's does not vanish at a point of (67.1), its continuity proves it different from zero in a region like (67.1) and contained in (67.1).

[23] Other proofs of the convergence will be found in [16], [12], [21], and [17].

Let now the finite set of f's be of class 2 in the same region (67.1). The quotient field described above will be interpreted as $\mathfrak{R}$. The proof that A_1 to A_5 are consistent has already been indicated in §12.

In the same way, the f's and their first partial derivatives give rise to a quotient field, every member of which is of class one in some region (67.1). This field can be made the $\mathfrak{R}'$ of Chapter IV by putting

$$\delta_i = \frac{\partial}{\partial x^i}. \tag{67.2}$$

Finally, we take $\mathfrak{R}''$ as the quotient field of the f's and their partial derivatives of order not exceeding two. Every member of $\mathfrak{R}''$ then represents a function continuous in some region (67.1).

The symbol

$$\delta_i \delta_j = \frac{\partial^2}{\partial x^i \partial x^j}$$

applied to any symbol of $\mathfrak{R}$ gives a continuous symbol of $\mathfrak{R}''$. Hence [8, I, 42] we have

$$\frac{\partial^2}{\partial x^i \partial x^j} - \frac{\partial^2}{\partial x^j \partial x^i} = 0, \tag{67.3}$$

so that relation (20.8) and assumption D_2 are satisfied. At the same time, the interpretation making D_3 valid should be clear.

68. **The integration process.** By the theorem [8, I, 227] on the integration of a function depending on parameters we know the existence of a $\sigma^i a$ defined by

$$\sigma^i a = \int_{x_0^i}^{x^i} a(x^1, \cdots, x^{i-1}, t, x^{i+1}, \cdots, x^n)\, dt,$$

which is of class two in a region together with the finite set of f's from which a is compounded.

Let $\mathfrak{R}$ be extended to include the totality of functions each of which can be obtained by applying to the f's a finite number of rational operations[24] and partial integrations. Any function whose symbol is in $\mathfrak{R}$ is of class 2 in a subregion of that admitted by the f's. Hence assumption I_1 is satisfied if $\mathfrak{R}$ is given the above interpretation.

Assumption I_2 is readily seen to be satisfied.

We shall next show that I_3 is satisfied. We may suppose that the pfaffian system is given by a finite set of linear forms, whose coefficients are of class 2 in a region (67.1). The coefficients serve, as the f's in §67, to define the $\mathfrak{R}$. There is a region (67.1) in which the coefficients of a normalized form of the basis are of class 2, that is, the basis can be written

$$\omega^\alpha = dx^\alpha + B^\alpha dx^n \qquad (\alpha = 1, \cdots, n-1), \tag{68.1}$$

where the B's have continuous second derivatives.

[24] We admit only rational combinations whose denominators are not identically zero in (67.1).

Let us seek to determine $d\varphi$ so that it is a differential of the pfaffian system. The condition $\omega^1 \cdots \omega^{n-1} d\varphi = 0$ gives

$$\frac{\partial \varphi}{\partial x^n} = B^\alpha \frac{\partial \varphi}{\partial x^\alpha}. \tag{68.2}$$

To show the existence of φ, we shall employ the method of successive approximations [14, 368].

The continuity of the B's shows the existence of a positive constant M such that

$$| B^\alpha | < M$$

in a region

$$| x^\alpha - x_0^\alpha | \leqq MR, \qquad | x^n - x_0^n | \leqq S, \tag{68.3}$$

interior to (67.1). For the moment we shall not need to use the fact that the B's have continuous first derivatives, but only the less stringent Lipschitz condition implied by it, namely,

$$| B^\alpha(x^1, \cdots, x^{n-1}, x^n) - B^\alpha(y^1, \cdots, y^{n-1}, x^n) | < A_\alpha | x^\alpha - y^\alpha |, \tag{68.4}$$

where the A's are positive constants and $(x^1, \cdots, x^{n-1}, x^n)$, $(y^1, \cdots, y^{n-1}, x^n)$ are any two points of region (68.3) having the same x^n.

Consider the recurrence formulas

$$x_m^\alpha = x_0^\alpha - \int_{x_0^n}^{x^n} B^\alpha(x_{m-1}^1, \cdots, x_{m-1}^{n-1}, t)\, dt. \tag{68.5}$$

If we assume

$$| x_{m-1}^\alpha - x_0^\alpha | \leqq MR, \qquad | x_{m-1}^\alpha - x_{m-2}^\alpha | < \frac{MA^{m-2} | x^n - x_0^n |^{m-1}}{(m-1)!}, \tag{68.6}$$

where $A = A_1 + \cdots + A_{n-1}$, we readily deduce from (68.4) and (68.5) that formulas (68.6) also hold when $m - 1$ is replaced by m. As a starting point for the induction there is the case $m = 2$ for which the formulas are readily seen to hold. Hence (68.6) are satisfied by (68.5) for all values of $m \geqq 2$.

The second relation of (68.6) implies

$$| x_{m-1}^\alpha - x_{m-2}^\alpha | < M(AS)^{m-1}/A(m-1)!.$$

Since the right member of this inequality is the general term of a convergent series of constants, each series with general term $x_m^\alpha - x_{m-1}^\alpha$ converges uniformly. Each sequence x_m^α therefore converges uniformly to a limit x^α. Because of the first of (68.6), the point x lies in (68.3) provided x^n is restricted by the second relation of (68.3). Hence by the Lipschitz condition

$$| B^\alpha(x^1, \cdots, x^{n-1}, x^n) - B^\alpha(x_m^1, \cdots, x_m^{n-1}, x^n) | < \epsilon$$

whenever $| x^\alpha - x_m^\alpha | < \epsilon$ and $| x^n - x_0^n | < S$. The integrands in (68.5) therefore converge uniformly to $B^\alpha(x)$. Passage to the limit in (68.5) is accordingly legitimate and gives

$$x^\alpha = x_0^\alpha - \int_{x_0^n}^{x^n} B^\alpha(x^1, \cdots, x^{n-1}, t)\, dt. \tag{68.7}$$

The x^α above are functions of x_0^α and x^n. They have continuous first derivatives with respect to those variables. We prove this first for the approximating functions x_m^α. Using now the continuity of the first derivatives of the B's and applying induction to (68.5), we readily deduce the result for x_m^α. The existence of the second derivatives of the B's enables us to formulate a Lipschitz condition for their first derivatives. By paralleling the above discussion we prove that the sequences $\partial x_m^\alpha / \partial x_0^\beta$ converge uniformly. Hence their limits are not only continuous but are also the derivatives of the limiting functions x [8, I, 71].

That dx_0^α furnish the desired $n - 1$ differentials of the pfaffian system follows by the argument already applied to (66.2). We have, therefore, proved that I_3 is satisfied.

69. **The analytic case.** If $\mathfrak{R}$ contains the field of complex numbers, A_7 and A_8 are valid in the subring of rational integers. Moreover, by the fundamental theorem of algebra the subfield of constants is closed.

The so-called Weierstrass preparation theorem is as follows.

THEOREM 69.1. *If $F(x, y) = F(x_1, \cdots, x_n, y)$ is holomorphic about $x = y = 0$ and $F(0, 0) = 0$, then*

$$\begin{aligned} F(x, y) &= [y^m + a_1(x)y^{m-1} + \cdots + a_{m-1}(x)y + a_m(x)]\, \varphi_1(x, y) \\ &= f(x, y)\, \varphi_1(x, y), \end{aligned} \tag{69.1}$$

where the a's are holomorphic about $x = 0$ and vanish for $x = 0$ and $\varphi_1(0, 0) \neq 0$.

This theorem is proved in [13, 86].[25] We shall establish the following consequence of it.

THEOREM 69.2. *A denumerably infinite determined system S of equations holomorphic in a region D of the form (67.1) is equivalent in a subregion to a system having at most one equation of each ordinal.*

The result is true for one variable. If the equations are F_i and a coefficient in an expansion is different from zero and the numerical determination is taken as zero, we have

$$F_i = x^{\alpha_i}(a_i + b_i x + \cdots) \qquad (\alpha_i > 0,\ a_i \neq 0).$$

[25] Note that assumption W is *not* the generalization of Osgood's theorem on p. 89, proved false by an example on pp. 90–91. Assumption W merely states that representation (54.1) is valid for some numerical determination, but not necessarily for all, i.e., vanishing and analyticity at the origin are not sufficient conditions for representation in the form (54.1) about the origin. Osgood's example shows the existence of what we have termed *singular zeros* in the preface.

If all its coefficients are zero, the F_i can be omitted. If a subregion in which $a_i + b_i x + \cdots$ has no zero is denoted by D_i, in any D_i the system is equivalent to x.

In the case of $n + 1$ variables, write the equations of ordinal $n + 1$

$$F_i = a_{ij}(x_1, \cdots, x_n)y^j \qquad (j = 0, 1, \cdots, \infty),$$

and assume the theorem for n variables. If every coefficient of every F_i vanishes for every numerical determination of S in D, the equations of ordinal $n + 1$ can be replaced by the denumerable system of equations a_{ij}, whose ordinal does not exceed n. The theorem is then true by hypothesis.

Suppose, therefore, that S has a numerical determination $x = y = 0$ for which a coefficient of $F(x, y)$, written as a series in y, does not vanish. Theorem 69.1 is then applicable. From considerations of continuity it follows that $\varphi_1(x, y) \neq 0$ in a subregion D_1 of D. If the discriminant of the polynomial $f(x, y)$ in (69.1) is identically zero in D_1, f and f' have a common factor in D_1. We suppose that common factor removed, if originally present. Then for some x's arbitrarily near zero (but at least one not zero) the discriminant is not zero. Substituting such a set of x's in $f(x, y)$ we have a polynomial in y with m distinct roots approaching zero with the x's. If the x's are given properly chosen small values, therefore, there are m y's which, taken with the x's, give a numerical determination for S in D for which the discriminant does not vanish. Let these points be $P_1 = (\xi_1, \cdots, \xi_n, \eta_1), \cdots, P_m = (\xi_1, \cdots, \xi_n, \eta_m)$. D_1 thus has a subregion D_2 containing, say, P_1 but none of the other P's. F has a unique root in D_2 when the values of $x_1, \cdots, x_n$ have been assigned. Hence F is equivalent in D_2 to $y - g(x)$, where g is holomorphic in D_2.

If $g(x)$ is substituted in every equation F_i for y, those equations change into equations holomorphic[26] in D_2 and having ordinal not exceeding n. Hence the theorem is true for $n + 1$ variables.

Assumption W is obviously an immediate consequence of the theorem just proved, if $\mathfrak{R} = \mathfrak{R}' = \mathfrak{R}''$ is the set of analytic functions.

Assumption Z is valid because of continuity.

The implicit function Theorem 55.2 is, of course, true for the analytic case [13, 12].

Consider a standard system $F_{r+1}, \cdots, F_l$, whose leaders are respectively $y_{r+1}, \cdots, y_l$. It is clear from the proof of Theorem 69.2 that *in a properly chosen region the system can be solved for* $y_{r+1}, \cdots, y_l$. It is also clear that E_0 is valid.

70. **Proof of E for the analytic case.** Consider a normal system, solved for its leaders and with right members holomorphic at the numerical determination, which we shall as usual suppose to consist of zeros. In addition, we shall assume that the same is true of the initial determination. That this does not restrict the generality follows from Theorem 47.1.

[26] A proof of this will be found in Riquier's treatise [16, 92].

Let the system S be

$$\frac{\partial z_\alpha}{\partial m} = f_\alpha(x, D), \qquad I = 0, \tag{70.1}$$

where m is a monomial in the independent variables, where the D represents parametric derivatives, possibly including the unknowns z, and where the f's represent functions holomorphic about $x = D = 0$.

If we set

$$z_\alpha = z_\alpha^* + \frac{m}{i_1! \cdots i_n!} f_\alpha(0, 0),$$

where $i_1 \cdots i_n$ is the index of m, system S becomes a system S^* in the unknowns z^*. Every D is unaltered or has added to it a function of x vanishing for $x = 0$. Hence $x = 0$, $D = 0$ implies $x = 0$, $D^* = 0$. On the other hand, $\partial z_\alpha/\partial m = \partial z_\alpha^*/\partial m + f_\alpha(0, 0)$. System S^*, therefore, has right members vanishing for the numerical determination. We accordingly may assume that this transformation has been performed and that the system (70.1) satisfies

$$f_\alpha(0, 0) = 0.$$

Each set of monomials M_α (§36) consists of a single monomial and is complete.

From two properties of holomorphic functions it follows that we may differentiate term by term and substitute one series in another.

If the prolonged systems are formed by differentiation, the principal derivatives can be eliminated completely (even from the original system) because the equations are linear in their leaders. (The process of §45 in general only reduces the equations and does not perform a complete elimination as here.) We shall denote the totality of the equations so obtained by S^∞, in keeping with the notation previously employed for prolonged systems (§44). Clearly the following useful statement is true.

THEOREM 70.1. *The principal derivatives of system* (70.1) *are polynomials with rational integers for coefficients and a finite number of the partial derivatives of the f's and the parametric derivatives for indeterminates.*

We wish next to determine the coefficients of the expansions for the solution z_α. If

$$z_\alpha = a_{\alpha i_1 i_2 \cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \qquad (i\text{'s} = 0, 1, \cdots, \infty),$$

then

$$a_{\alpha i_1 i_2 \cdots i_n} = \frac{1}{i_1!\, i_2! \cdots i_n!} \left(\frac{\partial^{i_1 + i_2 + \cdots + i_n} z_\alpha}{\partial x_1^{i_1}\, \partial x_2^{i_2} \cdots \partial x_n^{i_n}} \right)_{x=0}. \tag{70.2}$$

We shall call the coefficient *principal* or *parametric* according as the corresponding derivative in (70.2) is principal or parametric. The principal derivatives correspond to the multiples of M_α and the parametric to non-multiples of M_α.

From (47.2), (24.12) and Theorem 36.2 we conclude that every parametric coefficient of a solution is zero. Since equations like (70.2) also hold for the f's, evaluation of S^∞ for $x = 0$ accordingly gives

THEOREM 70.2. *The principal coefficients in a solution of* (70.1) *are polynomials in the coefficients of the right members with positive rational integers for coefficients. The parametric coefficients are zero.*

Expansions are thus uniquely determined. They give the z's, if a solution exists. We wish to prove next that the expansions converge in a region of the type (67.1) and consequently define functions holomorphic in that region. The convergence will be proved by the method of *dominant functions*, whose gist is as follows. Consider a system arising from (70.1) when the unknowns are replaced by Z:

$$\frac{\partial Z_\alpha}{\partial m} = F_\alpha(x, D), \qquad I = 0. \tag{70.3}$$

The coefficients in (70.1) and (70.3) are in an obvious one-to-one correspondence, and this correspondence extends to the coefficients of the solutions, if they exist. If the f's in (70.1) are given and we can determine F_α's so that every coefficient in an F_α is non-negative and is not less than the modulus of the corresponding coefficient in f_α, system (70.3) is said to *dominate* (70.1).

From Theorem 70.2 it is clear that *the coefficients of any solution of a dominant system are non-negative.*[27]

The polynomials which give the principal coefficients of the general system of type (70.1) can be used to evaluate the principal coefficients of z_α or Z_α. If in the polynomial for a coefficient c of z_α we substitute for every coefficient of an f_α its modulus, we get *a non-negative number c' which is not less than the modulus of c* because

$$|A + B| \leqq |A| + |B|, \qquad |AB| = |A|\,|B|.$$

Since (70.3) dominates (70.1), in addition *c' does not exceed the corresponding coefficient of Z_α.* If the series for Z_α converges absolutely at any point x, that for z_α will converge at the same point.

Our task is therefore to show how to pick a dominant system *which has a solution holomorphic at $x = 0$*, when we are given a system (70.1).

Let the functions f_α be holomorphic in the region

$$|x| \leqq \rho, \qquad |D| \leqq \rho. \tag{70.4}$$

[27] It is worth noting that this follows from the two facts: (i) (70.3) dominates (70.1); (ii) (70.3) has a solution. The proofs that the solution of (70.3) has non-negative coefficients customarily given are verifications rendered unnecessary by the remark in the text. In addition, it is nowhere necessary in the convergence proof to use the fact that the coefficients are non-negative.

The series $f_\alpha(\rho, \rho)$ are then absolutely convergent. Let M be an upper bound to the terms of their series of absolute values. The expansions of

$$F_\alpha = \frac{M}{1 - \dfrac{\Sigma x + \Sigma D}{\rho}} - M, \tag{70.5}$$

where ΣD is the sum of all parametric derivatives preceding the left member, by (say) the multinomial theorem are convergent in a region like (70.4) about the origin. The coefficient of any monomial in x, D of degree $m \neq 0$ in the expansion is $pM\rho^{-m}$, where p is a positive rational integer. The modulus of the corresponding term in f_α for $x = \rho$, $D = \rho$, is $c\rho^m$ and satisfies $c\rho^m < M$. Hence $c < M\rho^{-m} < pM\rho^{-m}$, and the F's dominate the f's.

Put $y = \Sigma x$ and seek a solution of the dominant system depending on y alone. Since

$$\frac{d}{dy} = \frac{\partial}{\partial x_1} = \cdots = \frac{\partial}{\partial x_n}$$

when the function to be differentiated has that nature, the system becomes

$$\frac{d^{k_\alpha} Z_\alpha}{dy^{k_\alpha}} = \frac{M}{1 - \dfrac{y + \Sigma D}{\rho}}, \tag{70.6}$$

and is no longer in solved form because the left members occur in the ΣD. In order to proceed, we find it convenient to make the assumption: k_α is the maximum order of a derivative of Z_α appearing in the system.

If (70.6) does not have this property, it can be replaced by an equivalent system which does, as will now be shown. The equation $A = 0$ is equivalent to $dA/dx = 0$, $A(0) = 0$. Hence any equation in (70.6) can be replaced by the result of differentiating it with respect to x provided the new system is regarded as determined, the new element in the initial determination being given by $A(0) = 0$.

Let the variable y be given a cote of the form $(1, \cdots)$. The first components of the cotes for the y-derivatives are then the same as those for the x-derivatives from which they arose.

Let Z_{1k} be the last left member in (70.6). The order of (70.6) in Z_1 is then easily seen to be k.

Let Z_{2q} be the next to the last left member in (70.6). Let (70.6) be of order l in Z_2. Replace the equation with left member Z_{2q} by its $(l - q)$th derivative, as indicated above. This leaves the order in Z_1 unaltered. For if Z_{1p} of cote $(c, \cdots)$ occurs in the right member of Z_{2l} of cote $(b, \cdots)$ and Z_{1k} has cote $(a, \cdots)$, then $Z_{1k} \geqq Z_{2l}$ implies $a \geqq b \geqq c$, whence $k \geqq p$.

Exactly the same argument shows that replacing the left member third from the last Z_{3r} by Z_{3s}, where s is the order of the system in Z_3, leaves its order in Z_1, Z_2 unaltered. And so on. Without loss of generality we may therefore make the assumption as stated.

Next we shall show that (70.6) defines

$$\frac{d^{k_\alpha} Z_\alpha}{dy^{k_\alpha}}$$

as functions holomorphic at the origin and having non-negative coefficients. We shall denote these derivatives by P_α and reserve D for the others. The equations on P_α can be written as

$$(\delta_{\alpha\beta} - A_{\alpha\beta})P_\beta = G_\alpha(x, D, P), \tag{70.7}$$

where the coefficients of the A's and G's are non-negative, the terms in G_α are at least of degree two in the P's, and δ is the Kronecker delta. Equations (70.7) have a numerical determination consisting of zeros.

Denote the constant term in $A_{\alpha\beta}$ by $a_{\alpha\beta}$. We wish next to show how the dominant system can be modified so that the new $\bar{a}_{\alpha\beta}$ satisfy

$$\bar{a}_{\alpha\alpha} < 1, \qquad \sum_\beta \bar{a}_{\alpha\beta} < 1 \qquad (\beta \neq \alpha), \tag{70.8}$$

there being no summation in the first inequality.

If x_i, z_α are replaced in (70.5) by $x_i\xi_i$, $z_\alpha\zeta_\alpha$, respectively, where $1 < \xi_i$, $1 < \zeta_\alpha$ and the repeated indices are not summed, the system will continue to dominate (70.1) because its coefficients are not decreased. The P's undergo [see (5.7)] the multiplication

$$P_\alpha \to \zeta_\alpha \xi_1^{i_1} \cdots \xi_n^{i_n} \bar{P}_\alpha,$$

where the monomial on the right varies for a given P_α and is determined by the derivative in (70.5) *from which P_α arose.*

Now $a_{\alpha\alpha}$ consists of the sum of the coefficients of the D's on the right of (70.5) which give rise to P_α on making the identification $d/dy = \partial/\partial x_i$. On dividing (70.7) by the positive monomial

$$\zeta_\alpha \xi_1^{i_1} \cdots \xi_n^{i_n}, \tag{70.9}$$

where $\alpha i_1 \ldots i_n$ is the index of the derivative on the *left* of (70.5) and comparing it with the corresponding new equation, we find

$$\bar{a}_{\alpha\alpha} < b_\alpha a_{\alpha\alpha},$$

where b_α is the maximum of the quotient of monomial (70.9) by a monomial $\zeta_\beta \xi_1^{j_1} \cdots \xi_n^{j_n}$ corresponding to a D in the right member of (70.5).

In the same way we have

$$\bar{a}_{\alpha\beta} < c_\alpha a_{\alpha\beta} \qquad (\alpha \neq \beta),$$

where c_α is the maximum ratio of (70.9) to a monomial $\zeta_\beta \xi_1^{j_1} \cdots \xi_n^{j_n}$ corresponding to a derivative on the right of (70.5) which gives rise to $P_\beta \neq P_\alpha$.

If we let the ϵ of Theorem 5.2 be small enough to satisfy all the conditions

$$\epsilon < \frac{1}{a_{\alpha\alpha}}, \qquad \epsilon < \frac{1}{\sum_{\beta} a_{\alpha\beta}},$$

the conditions (70.8) are satisfied.

Suppose, therefore, that the original system satisfies

$$a_{\alpha\alpha} < 1, \qquad \sum_{\beta} a_{\alpha\beta} < 1. \tag{70.10}$$

The functional determinant of (70.7) with respect to the P's evaluated for the numerical determination is

$$| \delta_{\alpha\beta} - a_{\alpha\beta} |. \tag{70.11}$$

Elementary methods [21, 293] show that it is positive and the algebraic complements of its elements are non-negative.

From the first of these facts and the implicit function theorem it follows that (70.7) can be solved for P_α as a set of functions of x, D holomorphic in the neighborhood of the numerical determination.

If (70.7) is differentiated with respect to x, D and evaluated for the numerical determination, the coefficients of P_α of highest order have precisely the determinant (70.11). By an induction and an appeal to the second of the properties quoted above for determinant (70.11) it can, therefore, be proved that the coefficients of the expansions P_α are all non-negative.

Cauchy's theorem [8, II, 368] for the case of ordinary equations can be applied to the solved form of (70.6) to show the existence of a unique holomorphic solution Z_α.

The proof that the expansions for z_α converge is accordingly complete.

The expansions substituted in

$$\frac{\partial z_\alpha}{\partial m} - F_\alpha(x, D) \tag{70.12}$$

give holomorphic functions, all of whose derivatives vanish for the numerical determination because of the way in which the coefficients were determined. Hence (70.12) are zero, and assumption E is proved.

CHAPTER XI

ILLUSTRATIVE EXAMPLES

This final chapter gives some examples illustrating the general theories developed in the preceding pages. We begin with an important application of the non-commutative law of multiplication.

71. **Non-commutative multiplication in integrals.** If a, b are real numbers such that $a \leqq b$ and $f(x)$ is a real function of the real variable x, under certain conditions, which we need not specify, there is defined as a limit a real number I, which depends on a, b, and f. The symbol

$$\int_a^b f(x)\,dx$$

is called the integral of f from a to b and its value is taken as $+I$, whereas the symbol

$$\int_b^a f(x)\,dx$$

is taken as $-I$. There is thus associated with the interval and the function a number $\pm I$, the sign chosen depending upon the order in which the end points (or boundary) of the interval are taken: if the sense of the boundary agrees with the positive sense on the line (the x-axis), the positive sign is chosen; if the sense of the boundary agrees with the negative sense on the x-axis, the negative sign is chosen.

Likewise, in the plane a function $f(x, y)$ and the area A enclosed by a curve C determine a number I. We suppose the curve *oriented*; that is, it has a definite sense of description. We then set

$$\int\int_A f(x, y)\,dx\,dy = \pm I, \tag{71.1}$$

the appropriate sign being that of the sense of description of A's boundary. The standard of comparison usually adopted is the rotation which carries the positive x-axis through 90° into the positive y-axis. If the opposite is chosen, however, it seems only natural to indicate this by writing

$$\int\int_A f(x, y)\,dy\,dx. \tag{71.2}$$

This A has value $\pm I$ according as the sense on the boundary of A agrees with that of a rotation of the positive y-axis through 90° into the positive x-axis. Consequently, if A and the sense of description of its boundary are fixed,

(71.2) and the left member of (71.1) are evidently equal and opposite. It is consistent with this fact to put

$$dx\,dy + dy\,dx = 0.$$

Hence Grassmann multiplication of differentials arises rather naturally.

The same notion can be used to define

$$\int_{R_n} f(x)\,dx_1 \cdots dx_n,$$

where R_n is an n-dimensional region with closed boundary R_{n-1}. This definition is not that usually[28] given in a first treatment of integration. It is customary, although rather illogical, to abandon the notion of an oriented boundary inherent in the definition of the ordinary one-dimensional integral.

The advantage in retaining the oriented boundary and in employing the Grassmann calculus can be illustrated as follows. The formula

$$\int_{R_n} \omega = \int_{R_{n-1}} \omega',$$

where R_n is a closed region, R_{n-1} its boundary, ω a symbolic differential form of degree p, and ω' the differential of ω, is a compact and general way of writing a useful relation, which has as many aliases (Green's, Riemann's, Gauss', Stokes', Ostrogradsky's) as it has disguises. In reality, it is nothing more than the first step in expressing a multiple integral as an iterated integral with one integration performed.

The formula for change of variables in a multiple integral becomes

$$\int F\,dx_1 \cdots dx_n = \int FJ\,d\bar{x}_1 \cdots d\bar{x}_n \tag{71.3}$$

without the absolute value signs needlessly and clumsily placed around the jacobian. The right member of (71.3), moreover, arises from the left by direct substitution. Thus the formal substitution

$$x = r\cos\theta, \qquad y = r\sin\theta$$

performed on

$$\iint dx\,dy$$

gives the polar formula

$$\iint r\,dr\,d\theta.$$

[28] J. Hadamard, *Cours d'Analyse*, Paris, 1927, vol. 1, p. 453, has pointed out the desirability of the present definition and attributed its conception to Méray.

72. **Reduction of quadratic forms in a Grassmann ring.** To illustrate the reduction of §17, consider the quadratic form

$$F = u_1u_3 - u_1u_4 + u_2u_3 - u_2u_4 + u_3u_5 - u_3u_6 + u_4u_5 - u_4u_6,$$

and the linear form u_1. Obviously, $u_1F^3 = 0$. To find u_1F^2 we ignore the first two terms of F and get

$$u_1F^2 = 4u_1(u_2u_3u_4u_5 - u_2u_3u_4u_6),$$

which can be factored by inspection thus:

$$u_1F^2 = 4u_1u_2u_3u_4(u_5 - u_6).$$

The right member is divisible by $u_2 + u_3$ because its product by $u_2 + u_3$ is zero. Hence we get also

$$u_1F^2 = 4u_1(u_2 + u_3)u_3u_4(u_5 - u_6).$$

Choose u_3 as the second linear form:

$$u_1u_3F = u_1u_3u_4(u_2 + u_5 - u_6).$$

Choose u_4 as the third linear form. Then $u_1u_3u_4F = 0$, and

$$F = u_1v_1 + u_3v_3 + u_4v_4.$$

The v's are by no means uniquely determined. One set of values, obtained by inspection, is exhibited in

$$F = u_1(u_3 - u_4) + u_3(-u_2 + u_5 - u_6) + u_4(u_2 + u_5 - u_6).$$

If no linear form is assumed, we find $F^3 = 0$ and

$$F^2 = 4(u_1 + u_2)u_3u_4(u_5 - u_6).$$

Choose $u_1 + u_2$ as the first linear form:

$$(u_1 + u_2)F = (u_1 + u_2)(u_3 + u_4)(u_5 - u_6).$$

Choose $u_3 + u_4$ as the second:

$$F = (u_1 + u_2)v_1 + (u_3 + u_4)v_2.$$

v_2 is obviously a factor of $(u_1 + u_2)F$. An arbitrary constant times $(u_3 + u_4)$ can of course be added to it and $(u_1 + u_2)$ if present can be removed by modifying v_1. Hence choose $v_2 = u_5 - u_6$. Substitution enables us to determine v_1 so that a canonical form of F is

$$F = (u_1 + u_2)(u_3 - u_4) + (u_3 + u_4)(u_5 - u_6).$$

An alternative way [3, 53–54] of finding a canonical form for F is as follows. If $a_{12} \neq 0$, the form

$$\varphi = F - \frac{1}{2a_{12}}\frac{\partial F}{\partial u_1}\frac{\partial F}{\partial u_2}$$

does not involve u_1 or u_2, as can be verified by taking its derivatives according to the method of §10. In the present case, $a_{13} = \frac{1}{2} \neq 0$.

$$\frac{\partial F}{\partial u_1} = u_3 - u_4, \qquad \frac{\partial F}{\partial u_3} = -u_1 - u_2 + u_5 - u_6,$$

$$\varphi = F - \frac{\partial F}{\partial u_1}\frac{\partial F}{\partial u_3} = 2u_4 u_5 - 2u_4 u_6.$$

Since the new $a_{45} = 1 \neq 0$, we repeat the process and find

$$\frac{\partial \varphi}{\partial u_4} = 2(u_5 - u_6), \qquad \frac{\partial \varphi}{\partial u_5} = -2u_4,$$

$$\varphi - \frac{1}{2}\frac{\partial \varphi}{\partial u_4}\frac{\partial \varphi}{\partial u_5} = 0.$$

Hence

$$F = \frac{\partial F}{\partial u_1}\frac{\partial F}{\partial u_3} + \frac{1}{2}\frac{\partial \varphi}{\partial u_4}\frac{\partial \varphi}{\partial u_5} = (u_3 - u_4)(-u_1 - u_2 + u_5 - u_6) + 2u_4(u_5 - u_6)$$

is the canonical form. It is easily transformed into that previously obtained.

73. **Reduction of pfaffian form of even class or of pfaffian equation to canonical form** (§27).

$$\begin{aligned}
\omega &= 2x_5\,dx_1 + (x_4 + x_5)dx_2 + 2x_4\,dx_3 + (x_5 - x_4)dx_4,\\
\omega' &= -2dx_1\,dx_5 - dx_2\,dx_4 - dx_2\,dx_5 - 2dx_3\,dx_4 - dx_4\,dx_5,\\
\omega\omega' &= -2x_5\,dx_1\,dx_2\,dx_4 + 2x_4\,dx_1\,dx_2\,dx_5\\
&\quad -4x_5\,dx_1\,dx_3\,dx_4 + 4x_4\,dx_1\,dx_3\,dx_5\\
&\quad -2x_4\,dx_1\,dx_4\,dx_5 - 2x_5\,dx_2\,dx_3\,dx_4\\
&\quad +2x_4\,dx_2\,dx_3\,dx_5 - 2x_4\,dx_2\,dx_4\,dx_5\\
&\quad -2x_4\,dx_3\,dx_4\,dx_5,
\end{aligned}$$

$$\omega'^2 \neq 0, \; \omega\omega'^2 = 0.$$

Hence $\rho = 2$, $\omega_{\rho-1} = \omega\omega'$. Next compute the associated set (§14). The marks of differentiation are indicated at the left.

12	$-2x_5\,dx_4 + 2x_4\,dx_5$
13	$-4x_5\,dx_4 + 4x_4\,dx_5$
14	$2x_5\,dx_2 + 4x_5\,dx_3 - 2x_4\,dx_5$
15	$-2x_4\,dx_2 - 4x_4\,dx_3 + 2x_4\,dx_4$
23	$-2x_5\,dx_4 + 2x_4\,dx_5$
24	$-2x_5\,dx_1 + 2x_5\,dx_3 - 2x_4\,dx_5$
25	$2x_4\,dx_1 - 2x_4\,dx_3 + 2x_4\,dx_4$

34 $\qquad -4x_5\,dx_1 - 2x_5\,dx_2 - 2x_4\,dx_5$

35 $\qquad 4x_4\,dx_1 + 2x_4\,dx_2 + 2x_4\,dx_4$

45 $\qquad -2x_4\,dx_1 - 2x_4\,dx_2 - 2x_4\,dx_3$

The combination 24–34 is $2x_5(dx_1 + dx_2 + dx_3)$. Hence we may set

$$g^1 = dx_1 + dx_2 + dx_3\,.$$

$$\omega_{\rho-2}g^1 = (x_5 - x_4)\,[dx_1\,dx_2 + 2dx_1\,dx_3 - dx_1\,dx_4 + dx_2\,dx_3 - dx_2\,dx_4 - dx_3\,dx_4]. \tag{73.1}$$

Form the associated set, after discarding the factor $(x_5 - x_4)$.

1 $\qquad dx_2 + 2dx_3 - dx_4$

2 $\qquad -dx_1 + dx_3 - dx_4$

3 $\qquad -2dx_1 - dx_2 - dx_4$

4 $\qquad dx_1 + dx_2 + dx_3$

Take g^2 as the first of these.

$$G = g^1g^2 = [dx_1\,dx_2 + 2dx_1\,dx_3 - dx_1\,dx_4 + dx_2\,dx_3 - dx_2\,dx_4 - dx_3\,dx_4]$$

From (27.9), in which $\rho + 1$ is to be interpreted as the present ρ,

$$\omega g^2 = f_1G, \qquad \omega g^1 = -f_2G.$$

Comparison of the coefficient of $dx_1\,dx_2$ in the second of these with (73.1) gives $f_2 = x_4 - x_5$. Similarly, the first gives $f_1 = 2x_5$. Hence a canonical form is

$$\omega = 2x_5(dx_1 + dx_2 + dx_3) + (x_4 - x_5)(dx_2 + 2dx_3 - dx_4),$$

a result easily checked with the original ω.

The reader should perform the reduction, using a different choice of solutions.

74. Reduction of pfaffian form of odd class.

$$\begin{aligned}
\omega &= (2x_5 + x_2)dx_1 + (x_1 + x_4 + x_5)dx_2 + 2x_4\,dx_3 + (x_5 - x_4)dx_4\\
\omega' &= -2dx_1\,dx_5 - dx_2\,dx_4 - dx_2\,dx_5 - 2dx_3\,dx_4 - dx_4\,dx_5\\
\omega'^2 &= 4dx_1\,dx_2\,dx_4\,dx_5 + 8dx_1\,dx_3\,dx_4\,dx_5 + 4dx_2\,dx_3\,dx_4\,dx_5\\
\omega\omega'^2 &= 4(-2x_1 + x_2)dx_1\,dx_2\,dx_3\,dx_4\,dx_5\\
\omega'^3 &= 0.
\end{aligned}$$

Hence $\rho = 2$, and the class is 5. In writing equation (27.5) we may ignore the terms in dx_4, dx_5 in z' because ω'^2 is divisible by $dx_4\,dx_5$. The coefficient of $dx_1\,dx_2\,dx_3\,dx_4\,dx_5$ equated to zero gives

$$\frac{\partial z}{\partial x_1} - 2\frac{\partial z}{\partial x_2} + \frac{\partial z}{\partial x_3} = -2x_1 + x_2\,,$$

a solution of which is $z = x_1 x_2$. For this value of z, the form $\omega - z'$ is precisely the form ω in §73, and the reduction already given serves. Hence

$$\omega = d(x_1 x_2) + 2x_5(dx_1 + dx_2 + dx_3) + (x_4 - x_5)(dx_2 + 2dx_3 - dx_4).$$

75. **An absolute complete set of monomials** (§36). Employ the variables x, y, z instead of x_1, x_2, x_3.

M: xy, xz, z^2

	Monomial	Multipliers	Non-multipliers
M^*	xyz^2	x, y, z	
	yz^2	y, z	x
	xz^2	x, z	y
	xyz	x, y	z
	z^2	z	x, y
	xz	x	y, z
	xy	x, y	z
$\bar{M}$	yz	y	x, z
	z		x, y, z
	y	y	x, z
	x	x	y, z
	1		x, y, z

If f is the unknown whose complete set is M^*, the initial determination consists of

$$(75.1)\qquad \begin{aligned} I_1 &= f(0, 0, 0), \qquad I_2(x) = \left(\frac{\partial f}{\partial x}\right)_{y=z=0}, \qquad I_3(y) = \left(\frac{\partial f}{\partial y}\right)_{x=z=0}, \\ I_4 &= \left(\frac{\partial f}{\partial z}\right)_{x=y=z=0}, \qquad I_5(y) = \left(\frac{\partial^2 f}{\partial y \partial z}\right)_{x=z=0}, \end{aligned}$$

where I_1 and I_4 are constants and the other I's are functions of the variables as indicated.

In the analytic case, we may write an arbitrary function $f(x, y, z)$ as

$$(75.2)\qquad \begin{aligned} f(x, y, z) = {} & f_{000} + xf_{100}(x) + yf_{010}(y) + zf_{001} + yzf_{011}(y) \\ & + xyf_{110}(x, y) + xzf_{101}(x) + xyzf_{111}(x, y) \\ & + z^2 f_{002}(z) + xz^2 f_{102}(x, z) + yz^2 f_{012}(y, z) \\ & + xyz^2 f_{112}(x, y, z), \end{aligned}$$

where f_{ijk} are arbitrary functions of their arguments as indicated. The right member of (75.2) consists of expressions $x^i y^j z^k f_{ijk}$, with $x^i y^j z^k$ belonging to M^* or $\bar{M}$, and the arguments of f_{ijk} are the multipliers of $x^i y^j z^k$. Theorems 36.1 and 36.2 are nicely illustrated by the expansion (75.2). For a development of the notion of complete set from the Maclaurin series see [21, 286].

An important property of (75.2) is: *the result of differentiating with respect to m and then putting the non-multipliers of m equal to zero is the same whether performed on f or* $x^i y^j z^k f_{ijk}$. As a consequence of this, the initial determination (75.1) determines f_{000}, $f_{100}(x)$, $f_{010}(y)$, f_{001}, $f_{011}(y)$ in (75.2) and conversely. We have, for example,

$$I_1 = f(0, 0, 0), \qquad I_2(x) = \frac{\partial}{\partial x}[x f_{100}(x)]\Big|_{y=z=0}$$

$$= x\,\frac{\partial f_{100}(x)}{\partial x} + f_{100}(x).$$

Likewise, integration of the second of these gives

$$x f_{100}(x) = \int_0^x I_2(t)\,dt.$$

Since the right member vanishes for $x = 0$, the factor x can be divided out to give $f_{100}(x)$.

76. **A corresponding relative complete set** (§37). Adopt the order x, y, z for the variables so that xy, xz, z^2 are arranged according to increasing relative rank. The sets (§37) M_m are

$$\begin{aligned} M_1&: \quad xy \\ M_2&: \quad xz, \quad xyz \\ M_3&: \quad z^2,\ xz^2,\ yz^2,\ xyz^2. \end{aligned}$$

Since in M_2 the quotient xyz/xz is y which is multiplier for xyz, we omit xyz and give xz the additional multiplier y. In the same way, $xyz^2/z^2 = xy$ involves only multipliers of xyz^2. Hence we omit xyz^2 and give z^2 the multipliers x, y. Subsequently xz^2, yz^2 are omitted, without changing the multipliers of z^2, which already has all the variables for multipliers. Performing the corresponding operation on the absolute complementary set gives the following table.

	Monomial	Multipliers	Non-multipliers
	xy	x, y	z
M^*	xz	x, y	z
	z^2	x, y, z	
	1	x	y, z
$\bar{M}$	y	y	x, z
	z	y	x, z

The significance of this is brought out in the analytic case by an examination of the expansion

$$\begin{aligned} f(x, y, z) &= f_{000}(x) + y f_{010}(y) + z f_{001}(y) \\ &\quad + xy f_{110}(x, y) + xz f_{101}(x, y) + z^2 f_{002}(x, y, z), \end{aligned} \tag{76.1}$$

which arises from (75.2) by regrouping the terms, that is,

$$f_{000}(x) = f_{000} + xf_{100}(x), \qquad f_{001}(y) = f_{001} + {}_y f_{011}(y),$$

$$f_{101}(x, y) = f_{101}(x) + yf_{111}(x, y),$$

$$f_{002}(x, y, z) = f_{002}(z) + xf_{102}(x, z) + yf_{012}(y, z) + xyf_{112}(x, y, z).$$

The initial determination is

$$f(x, 0, 0), \qquad \left(\frac{\partial f}{\partial y}\right)_{x=z=0}, \qquad \left(\frac{\partial f}{\partial z}\right)_{x=z=0}. \tag{76.2}$$

The reader should treat similarly the set

M: $\qquad x^2y^2, \quad xz, \quad y^2z, \quad z^2.$

If the order of the variables is x, y, z, the relative case gives

Monomial	Multipliers	Non-multipliers
x^2y^2	x, y	z
xz	x	y, z
xyz	x	y, z
y^2z	x, y	z
z^2	x, y, z	
1	x	y, z
y	x	y, z
y^2	y	x, z
xy^2	y	x, z
z		x, y, z
yz		x, y, z

77. **Simple systems.** Let the indeterminates be x, y, z in the order indicated.
Example 1. van der Waerden [23, II, 9].

$$S = f + g, \qquad f = x^2 + xy, \qquad g = xy + y^2 + x + y.$$

$$R(f, g) = \begin{vmatrix} x & x^2 & 0 \\ 0 & x & x^2 \\ 1 & x+1 & x \end{vmatrix} = 0, \qquad R_1(f, g) = x.$$

The left factor is therefore $S + \bar{x}$.

$$\varphi = (x)y + (x^2) = f,$$

$$S + \bar{x} = \{xy + x^2, \bar{x}\}.$$

This is readily seen to be simple.

Consider next $S + x$. The common of f and x being x, f can be omitted. When g is reduced by x, the system becomes $y(y + 1), x$, which is simple. Hence

$$S = \{xy + x^2, \bar{x}\}\{y^2 + y, x\}.$$

Example 2.

$$S = f + g + h, \qquad f = z^2 + 2(y - x)z + y^2 - 2xy,$$

$$g = z^2 + (3y - x)z + 3xy - 2x^2, \qquad h = z^2 + (y - 2x)z - 2xy,$$

$$R(f, g) = \begin{vmatrix} 1 & 2y - 2x & y^2 - 2xy & 0 \\ 0 & 1 & 2y - 2x & y^2 - 2xy \\ 0 & 1 & 3y - x & 3xy - 2x^2 \\ 1 & 3y - x & 3xy - 2x^2 & 0 \end{vmatrix}$$

$$= -4y^4 + 20xy^3 - 28x^2y^2 + 12x^3y,$$

$$R_1 = x + y.$$

Consider the factor $S + R + \bar{R}_1$. The common of f and g is

$$\varphi = (x + y)z - y^2 + 5xy - 2x^2.$$

$$R(\varphi, h) = \begin{vmatrix} x + y & -y^2 + 5xy - 2x^2 & 0 \\ 0 & x + y & -y^2 + 5xy - 2x^2 \\ 1 & y - 2x & -2xy \end{vmatrix}$$

$$= 2y^4 - 18xy^3 + 30x^2y^2 - 14x^3y.$$

$$R_1(\varphi, h) = x + y.$$

We next wish the common of $R(f, g)$ and $R(\varphi, h)$. To avoid long computations, we employ an indirect method. Factorization by inspection gives

$$R(f, g) = -4y(y - x)^2(y - 3x),$$
$$R(\varphi, h) = 2y(y - x)^2(y - 7x).$$

Hence the common can be taken as $y(y - x)$.

We then have an equation and an inequation of ordinal 2. The resultant of the two is $-2x^2$. If $x \neq 0$, the inequation can be abandoned. If $x = 0$, the two are inconsistent. In the first case, φ must be reduced by means of $y^2 - xy$. We accordingly find

$$S + R(f, g) + R(\varphi, h) + \overline{R_1}(f, g) = \{(x + y)z + 4xy - 2x^2, y^2 - xy, \bar{x}\}.$$

To treat the factor for which

$$R_1(f, g) = x + y = 0,$$

it is perhaps easiest to reduce the equations of S by that equation before proceeding. We have then

$$z^2 - 4xz + 3x^2, \qquad z^2 - 4xz - 5x^2, \qquad z^2 - 3xz + 2x^2.$$

From the first two $x = 0$. Hence we have as the final factorization into simple systems

$$S = \{(x + y)z + 4xy - 2x^2, y^2 - y, xy, \bar{x}\} \ \{z, y, x\}.$$

From the above it should be clear that the calculations, which may be effected in various ways, may often be shortened by a little ingenuity.

The reader should work the same example, forming the resultant $R(f, h)$ first.

78. **Ordinary algebraic differential systems** (§§46, 49). Consider the system (Ritt [17, 13])

$$S = f + g, \qquad f = z_1^2 - 4z, \qquad g = z_2 - 2,$$

where there is a single unknown z and differentiation with respect to the single independent variable is denoted by a subscript: $z_1 = dz/dx$, etc. The complete set of monomials is x^2, x, of which the second has x for non-multiplier. The conditions for a standard system are obviously satisfied. In accordance with §46, we differentiate f with respect to its non-multiplier x and eliminate the leader in the result by means of g. Thus we have

$$\frac{df}{dx} = 2z_1z_2 - 4z_1,$$

and the condition of passivity is the equation

$$\frac{df}{dx} - 2z_1g. \tag{78.1}$$

Since it vanishes identically, the system is passive. The initial determination corresponds to the monomial 1 with no multiplier, that is, the initial determination is the value of z for $x = 0$, say c.

Let us now decompose S into normal systems. Since f has the non-multiplier x, if we put $(z_1)_0 = \zeta$, we have for the decomposition

$$S[0] = z_2 - 2, \qquad S[x] = \zeta^2 - 4c. \tag{78.2}$$

The proof of Theorem 51.1 shows that a solution of the first of these, whose z_1 reduces to ζ and whose z reduces to c for $x = 0$, is the unique solution, whose existence is stated by the theorem. That the solution of (78.2) satisfies f is readily seen by substituting in (78.1). The result is $df/dx = 0$, and since $f = 0$ for $x = 0$, we must have $f = 0$ under the substitution.

The reader should treat the system

$$S = f + g, \qquad f = z_2 - zz_1, \qquad g = xz_3 + z_1.$$

79. **Partial algebraic differential systems** (§46). Let there be two independent variables x, y and a single unknown z, and use the mongean notation $p = \partial z/\partial x$, $q = \partial z/\partial y$. Let the system be

$$f = p^2 - (x + y + 1)p + x + y, \qquad g = q^2 - (x + 1)q + x.$$

This is clearly a simple system. Employ the relative complete set for the order x, y. Differentiating f with respect to the multiplier y and g with respect to the non-multiplier x gives

$$
\begin{aligned}
&[2p - (x + y + 1)]s - p + 1, \\
&[2q - (x + 1)]s - q + 1.
\end{aligned} \tag{79.1}
$$

Elimination of s yields as the one passivity condition (46.3):

$$h = (1 - x)p + (x + y - 1)q - y. \tag{79.2}$$

We still must form the passivity condition for (79.2). Differentiation of (79.2) with respect to y and g with respect to y gives

$$(1 - x)s + (x + y - 1)t + q - 1,$$

$$(2q - x - 1)t.$$

Eliminate s, t from these and the second of (79.1):

$$
\begin{vmatrix}
1 - x & x + y - 1 & q - 1 \\
0 & 2q - x - 1 & 0 \\
2q - x - 1 & 0 & -q + 1
\end{vmatrix}. \tag{79.3}
$$

It is readily found that

$$R(g, h) = (1 - x)^2 f, \qquad R_1(g, h) = 0.$$

Hence $f + h$ implies g. If q follows p, it is seen to be a standard, passive system.

A numerical determination for $f + h$ is $(p, q, x, y) = (0, 0, 0, 0)$. An initial determination is $z(0) = 0$. The root of f corresponding to the numerical determination is $p = x + y$. The introduction of this in (79.2) gives $q = x$. The unique solution, which can be found by integrating a perfect differential (Theorem 21.1), is $z = \frac{1}{2}x^2 + xy$.

The only other numerical determination for f, if x, y are fixed as 0, 0, is $p = 1$. The corresponding root of f is $p = 1$ (it happens to be constant) and from (79.2) we have $q = 1$. The unique solution of the system reducing to zero for $x = y = 0$ is $z = x + y$.

The foregoing example illustrates the general process very well. It should be remarked, however, that except in the simplest cases the computations are apt to become very involved. No device which will simplify the expressions should be overlooked. In particular, if an equation f of S can be readily factored into $f = f_1 f_2$, the system should be written

$$S = (S - f + f_1)(S - f + f_1 + f_2).$$

The equations of the example above can be written

$$
\begin{aligned}
f &= f_1 f_2, & g &= g_1 g_2, & f_1 &= p - x - y, \\
f_2 &= p - 1, & g_1 &= q - x, & g_2 &= q - 1,
\end{aligned}
$$

and

$$S = (f_1 + g_1)(f_1 + g_2)(f_2 + g_1)(f_2 + g_2). \tag{79.4}$$

Consider $f_1 + g_2$. Differentiating g_2 with respect to the non-multiplier x gives s, which is the leader in $\partial f_1/\partial y = s - 1$. Elimination of s gives the passivity condition 1, so that $f_1 + g_2 = 1$. In similar fashion, $f_2 + g_1 = 1$. On the other hand, the other two factors are passive and give the solutions already obtained.

80. **Decomposition into normal systems** (§49). The system to be considered is in a single unknown f and has for leaders the derivatives corresponding to the complete set of the first example in §76.

The system $S_{[1]}$ has for leaders

$$\frac{\partial^2 \zeta_1}{\partial x \partial y}, \qquad \frac{\partial \zeta_2}{\partial x},$$

where

$$\zeta_1 = f(x, y, 0), \qquad \zeta_2 = \left(\frac{\partial f}{\partial z}\right)_{z=0}. \tag{80.1}$$

	Monomial	Multipliers	Non-multipliers
M_1^*	xy	x, y	
$\bar{M}_1$	1	x	y
	y	y	x
M_2^*	x	x	y
$\bar{M}_2$	1	y	x

The initial determinations are accordingly

$$\zeta_1(x, 0), \qquad \frac{\partial \zeta_1(0, y)}{\partial y}, \qquad \zeta_2(0, y),$$

and are given by (76.2).

The system $S_{[0]}$ has for leader

$$\frac{\partial^2 f}{\partial z^2},$$

and for initial determination

$$\left(\frac{\partial f}{\partial z}\right)_{z=0}, \qquad f(x, y, 0),$$

which is given by (80.1) when the ζ's have once been determined.

The parametric derivatives which may occur in the right members of S are evaluated as follows for $z = 0$:

$$\left(\frac{\partial f}{\partial x}\right)_{z=0} = \frac{\partial \zeta_1}{\partial x}, \qquad \left(\frac{\partial f}{\partial y}\right)_{z=0} = \frac{\partial \zeta_1}{\partial y}, \qquad \left(\frac{\partial f}{\partial z}\right)_{z=0} = \zeta_2,$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{z=0} = \frac{\partial^2 \zeta_1}{\partial x^2}, \qquad \left(\frac{\partial^2 f}{\partial y^2}\right)_{z=0} = \frac{\partial^2 \zeta_1}{\partial y^2},$$

$$\left(\frac{\partial^2 f}{\partial y \partial z}\right)_{z=0} = \frac{\partial \zeta_2}{\partial y}.$$

Hence $S_{[1]}$ is indeed a system in the unknowns ζ_1, ζ_2.

Let the reader reduce in similar fashion the system whose leaders correspond to yz, zx, xy.

81. **Singular integral varieties of a linear pfaffian equation** (§65). The pfaffian

$$\omega = (x + y)dx + zdy - (x + y)dz$$

has the singular integral surface

$$x + y - z = 0. \tag{81.1}$$

We have

$$\omega\omega' = (-x - y + z)dx\,dy\,dz.$$

Moreover, a transformation to canonical form

$$dz^* - y^*dx^*$$

is

$$x^* = x + 2y, \qquad y^* = -x - y + z, \qquad z^* = x^2 + y^2 - yz - zx + 2xy,$$

whose jacobian is $x + y - z$. The transformation to canonical form therefore fails on (81.1).

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